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Abstract

We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to allow for externalities, we establish that stable matchings exist when agent choices satisfy substitutability. Furthermore, we show that substitutability is a necessary condition for the existence of a stable matching in a maximal-domain sense and provide a characterization of substitutable choice functions. In addition, we establish novel comparative statics on externalities and show that the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, remain valid despite the presence of externalities even though the standard fixed-point techniques do not apply.

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Matching with Externalities

Marek Pycia and M. Bumin Yenmez*

August 2019

Abstract

We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to allow for externalities, we establish that stable matchings exist when agent choices satisfy substitutability. Furthermore, we show that substitutability is a necessary condition for the existence of a stable matching in a maximal-domain sense and provide a characterization of substitutable choice functions. In addition, we establish novel comparative statics on externalities and show that the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, remain valid despite the presence of externalities even though the standard fixed-point techniques do not apply.

1 Introduction

Externalities are present in many two-sided markets. For instance, couples in a labor market pool their resources as do partners in legal or consulting partnerships. As a result, the preferences of an agent may depend on the contracts signed by the partner(s). Likewise, a firm's

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hiring decisions are affected by how candidates compare to competitors' employees. Finally, because of technological requirements of interoperability, an agent's purchase decisions may change because of other agents' decisions.

In this paper, we incorporate externalities into the stable matching theory of Gale and Shapley (1962).[1] We refer to the two sides of the market as buyers and sellers. Each buyer-seller pair can sign many bilateral contracts. Furthermore, each agent is endowed with a choice function that selects a subset of contracts from any given set conditional on a reference set for the other agents. We build a theory of matching with externalities that both establishes new insights and extends to the settings with externalities some of the key insights of the classical theory without externalities, such as the existence of stable matchings and the role of the deferred acceptance (or cumulative offer) algorithm.

Our theory is built on a substitutes condition that extends the classical substitutes condition to the setting with externalities. We require that each agent rejects more contracts from any set than its subsets conditional on the same reference set (as in the classical substitutes condition) and also that each agent rejects more contracts from a set conditional on a reference set μ than the same set conditional on a reference set μ' such that μ reflects better market conditions than μ' for his side of the market. The idea of better market conditions extends the revealed preference idea of Blair (1988) to the setting with externalities. When there are no externalities, this substitutes condition reduces to the classical gross substitutes condition of Kelso and Crawford (1982).

We start by proposing an algorithm akin to the deferred acceptance algorithm for the setting with externalities, which may be important in potential market-design applications. In particular, the algorithm can also be viewed as a new auction that performs well in the presence of externalities. Since an agent's choice depends on others' contracts, we keep track not only of which contracts are available but also of the reference sets that agents on each side use to condition their choice. The construction requires care because after the reference set has changed an agent may want to go back to a contract that is already rejected. To ensure that this does not happen, we construct the initial reference sets in a preliminary phase of the algorithm. Relatedly, we cannot stop the algorithm as soon as the set of available contracts converge: we need to continue until the reference sets converge as well. Our construction of initial reference sets ensures that subsequent reference sets change in a monotonic way with

[1] Let us stress that even though we derive our results in a general many-to-many matching setting with contracts (cf. e.g. Hatfield and Milgrom, 2005, Klaus and Walzl, 2009, and Hatfield and Kominers, 2017), the results are new in all special instances of our setting, including many-to-one and one-to-one matching problems.

respect to the better market conditions preorder, thus ensuring that from some point on the reference sets belong to the same equivalence class. While these equivalence classes might consist of many matchings, we further show that the algorithm converges to one of them and never cycles among the members of the same equivalence class.

Our main results show that the algorithm always converges to a stable matching when choice functions satisfy substitutability (Theorem 2), and hence that stable matchings exist.[2] We also show that substitutability is necessary for the existence of a stable matching in a maximal-domain sense (Theorem 3), extending the insights of Hatfield and Milgrom (2005), Hatfield and Kojima (2010), and Hatfield and Kominers (2017) for the standard substitutability condition in a setting without externalities. Furthermore, we provide a characterization of substitutable choice functions (Theorem 1): a choice function satisfies the substitutes condition if and only if the choice from a set consists of the highest ranked contracts according to some ranking, where the set of allowed rankings is fixed for the choice function. This characterization is inspired by the decomposition result of Aizerman and Malishevski (1981) for the setting without externalities.[3]

In addition to the main results, we show that the presence of externalities in the choice behavior of agents on one side harms agents on this side and benefits agents on the other side provided the removal of externalities entails a contraction of the agents' choices (Theorem 7).[4] We also extend the classical theory of matching to the setting with externalities. In Section 5, we show that every stable matching is Pareto efficient (Theorem 5) and we analyze the existence of side-optimal stable matchings, that is, matchings that are maximal in the better market condition preorder. An optimal stable matching exists for side θ under the additional assumption that there exists a matching that reflects better market conditions than any other matching for side θ (Theorem 6). This additional assumption is satisfied trivially in settings without externalities, where the existence of side-optimal stable matchings was established by Gale and Shapley (1962) for the marriage problem.

[2] We focus on the classical short-sighted stability concept in which each agent assumes that other agents do not react to their choice. Our results, however, are applicable to many other stability concepts including far-sighted ones because we formulate the results in terms of agents' choice behavior and not in terms of their preferences. See Remark 1 of the previous version of our paper, which is available at `http://dx.doi.org/10.2139/ssrn.2475468`.

[3] For applications of such a decomposition result in settings without externalities see Chambers and Yenmez (2017).

[4] This result contains as an important special case the situation when an agent retires from the market (cf. Kelso and Crawford (1982) and Crawford (1991)). Simultaneously with our work, comparative statics in matching have been analyzed by Echenique and Yenmez (2015) and Kamada and Kojima (2019); they do not allow for externalities, and even in the setting without externalities our result remains new.

We also generalize the rural hospitals theorem (McVitie and Wilson, 1970; Roth, 1984; Hatfield and Milgrom, 2005), which states that each agent gets the same number of contracts in every stable matching in a many-to-one matching problem without externalities (in Appendix A). Our generalization allows different contracts to have different weights that may depend on the quantity, price, or quality of the contracts. For this purpose, we introduce a general law of aggregate demand. An agent's choice function satisfies *the law of aggregate demand* if the weight of contracts chosen from a set conditional on a reference set μ is greater than the weight of contracts chosen from a subset conditional on a reference set that has worse market conditions than μ. When there are no externalities, this law of aggregate demand reduces to the monotonicity condition of Fleiner (2003). We show that when choice functions satisfy the law of aggregate demand in addition to the aforementioned properties, all stable matchings have the same weight for every agent (Theorem 8).

To the best of our knowledge, our development of the substitutability condition and our results that go beyond existence have no forerunners in the literature analyzing externalities in the setting of Gale and Shapley (1962). We thus contribute to the matching literature by showing how one can incorporate externalities into standard models of matching, including matching with contracts, by offering new insights, and by showing that many of the results of the classical literature remain valid in the presence of externalities.

On the other hand, our existence result contributes to a rich literature analyzing the existence and nonexistence results in matching with externalities. In an early influential paper, Sasaki and Toda (1996) showed that stable one-to-one matchings need not exist. Their insight led the subsequent literature to take one of two routes: to modify the stability concept, or to impose assumptions on agents' preferences. Sasaki and Toda's seminal paper belongs to the first strand of literature. They focused on a weak stability concept that allows a pair of agents to block a matching only if they benefit from the block under all possible rematches of the remaining agents. They show that such weak stable matchings exist.[5] In contrast, our paper uses the standard stability concept of Gale and Shapley (1962) and the literature on matching without externalities. We guarantee the existence of stable matchings not by modifying the stability concept but by imposing assumptions on preferences in line with the standard approach of restricting attention to substitutable preferences.

[5]The rich subsequent literature, e.g., Chowdhury (2004); Hafalir (2008); Eriksson, Jansson, and Vetander (2011); Chen (2013); Gudmundsson and Habis (2017); Salgado-Torres (2011a,b)—maintained the focus on the existence question while refining Sasaki and Toda's weak stability concept by varying the degree to which the rematches of other agents penalize the blocking pair. Bodine-Baron, Lee, Chong, Hassibi, and Wierman (2011) analyze a related weak stability concept in a setting with peer effects.

The second strand of the literature analyzes the standard stability concept. Prior work in this second strand of the literature identified several assumptions under which stable matchings exist. Particular attention has been devoted to externalities among couples (Dutta and Massó, 1997; Klaus and Klijn, 2005; Kojima, Pathak, and Roth, 2013; Ashlagi, Braverman, and Hassidim, 2014), to peer effects among students matched to the same college (Dutta and Massó, 1997; Echenique and Yenmez, 2007; Pycia, 2012; İnal, 2015), and to student assignment problem with neighbors (Ashlagi and Shi, 2014; Dur and Wiseman, 2019). We are not restricting our attention to either of these types of externalities.

Our contribution on the existence of stable matchings is closest to the few papers that look at standard stability in the selected matching problems with externalities. Bando (2012; 2014) studies many-to-one matching allowing externalities in the choice behavior of firms (agents who match with potentially many agents on the other side) but not of workers; he further assumes that each firm's choice function depends on the matching of other firms only through the set of workers hired by other firms, and imposes several other elegant assumptions on firms' choice behavior. Under these assumptions, he proves the existence of stable matchings and analyzes the deferred acceptance algorithm. In his setting there is no need to keep track of the reference sets in the deferred acceptance algorithm (and hence no need for the preliminary phase that constructs the initial reference sets), and his algorithm terminates as soon as there are no rejections. In another related work, Teytelboym (2012) looks at externalities among agents in a component of a network and shows that a stable matching exists provided agents' preferences are aligned in the sense of Pycia (2012). Finally, Fisher and Hafalir (2016) consider a setting in which each agent cares only about the level of externality in the overall economy (such as pollution) and study the existence of stable matchings when there are such aggregate externalities. These papers provide the sufficient conditions for stability with externalities with the exception of Pycia (2012), who similarly to us—but within the confines of the college admission setting he studies—shows that his preference alignment condition is not only sufficient but also necessary in the maximal domain sense.[6]

[6] Pycia's alignment condition is neither implied by nor implies standard substitutability as discussed in his paper. Furthermore, his alignment condition cannot be satisfied in models with transfers the receiver of the transfer prefers a higher payment while the sender prefers a lower payment (cf. Pycia, 2008), and, in particular, the conjunction of standard substitutability and monotone externalities also do not imply his alignment condition. In a different setting with externalities, a restricted necessity result was provided by Ali (2016): assuming that all workers prefer to join firms with more employees, and all firms have responsive preferences with no capacity constraints, a stable matching exists if and only if there are at most two firms in the market. Uetake and Watanabe (2012) who provide an empirical analysis of firm mergers using a matching model with externalities, and Mumcu and Saglam (2010) who analyze when all matchings in the non-empty collection of top matchings are stable. Baccara, Imrohoroglu, Wilson, and Yariv (2012) analyze stable one-sided allocations with externalities.

Our work is also related to the exploration of efficiency in markets with externalities (see, e.g., Pigou (1932); Chade and Eeckhout (2019); Watson (2014)); while this literature focuses on efficiency, we focus on stability. Furthermore, one our examples (Example 6) shows the applicability of our results to the analysis of dynamic matching; for prior analysis of dynamic matching, see, e.g., Ünver (2010), Pycia (2012), Kurino (2014), and Kotowski (2015).

2 Examples

In this section, we provide two motivating examples. Additional motivating examples are provided in Appendix D and two illustrative examples are provided in Section 3.2.

As it is well known, the existence of a stable matching is not guaranteed in the presence of externalities (see Example 4). However, the examples in this section satisfy our substitutability condition that guarantees the existence. We come back to these examples after formally defining substitutability. For simplicity, we consider only one side of the market. One could model the other side in the same or a different way because we impose no assumptions relating the choice behavior of agents across sides.

Example 1. [Couples in a Local Labor Market][7] Agents on one side of the market represent workers and agents on the other side represent firms. Workers are either single or are members of exogenously married couples. The labor participation decision of a married man depends on the job of his wife: the better the job she has, the more selective he becomes. In other words, the outside option of not working becomes more preferred when a man's wife has a better job. We assume that there are no externalities for firms (whose preferences satisfy the standard substitutes condition) or the single workers.

This example can be generalized such that there are externalities for both married men and women. Furthermore, any two agents can be married, so we do not need a two-sided structure for the workers. ■

Our theory also applies to situations in which market participants care about the relative standings of their partners.

Example 2. [Relative Rankings in Hiring] Agents on one side of the market represent colleges and agents on the other side represent academics in a particular field. For each college i

Hatfield and Kominers (2015) study the existence of competitive equilibria in a multilateral matching setting with externalities. Leshno (2019), a work in progress, looks at large matching markets.

[7] We are grateful to Michael Ostrovsky for suggesting this example.

and each academic j the productivity of j at i is denoted by $\lambda(i, j) \geq 0$. For simplicity, assume that no two academics have the same productivity at a college.

Suppose that each college hires at most two academics in the field considered, and that it wants to hire at least one because of teaching needs and would like to hire a second academic only if their productivity is at least as high as the productivity of all academics in at least half of the other colleges. The interpretation is that a second academic is hired only if they are a "star" in the field.

Formally, the choice function $c_i(X_i|\mu)$ of college i is as follows: from choice set X_i, the college chooses the academic $j \in X_i$ with highest productivity $\lambda(i, j)$, and it chooses a second academic $j' \in X_i$ if, and only if, $\lambda(i, j')$ is greater than or equal to the productivity of all academics in at least half of the other colleges under matching μ. More generally, we can fix $k \in [0, 1]$ and assume that college i chooses a second academic $j' \in X_i$ if, and only if, $\lambda(i, j')$ is greater or equal than the productivity of academics in at least a fraction k of other colleges. ■

3 Model and Characterization of Substitutability

There is a finite set of agents $\mathcal{I}$ partitioned into buyers, $\mathcal{B}$, and sellers, $\mathcal{S}$, $\mathcal{B} \cup \mathcal{S} = \mathcal{I}$. The set of all agents on the same side with agent i is denoted as $\theta(i)$. Therefore, $\theta(i) = \mathcal{B}$ if i is a buyer and $\theta(i) = \mathcal{S}$ if i is a seller. With a slight abuse of notation, θ also denotes one side of the market, so $\theta \in \{\mathcal{B}, \mathcal{S}\}$. If θ is a side, then $-\theta$ is the other side, that is, $-\mathcal{B} \equiv \mathcal{S}$ and $-\mathcal{S} \equiv \mathcal{B}$. Agents interact with each other bilaterally through contracts. Each contract x specifies a buyer $b(x)$, a seller $s(x)$, and terms, which may include price, quantity, and quality. There exists a finite set of contracts $\mathcal{X}$. For any $X \subseteq \mathcal{X}$, X_i denotes the set of all contracts in X involving agent i, that is $X_i \equiv \{x \in X : i \in \{b(x), s(x)\}\}$. Similarly, X_{-i} denotes the set of all contracts not involving agent i, that is, $X_{-i} \equiv X \setminus X_i$.

Each agent i has a choice function c_i, where $c_i(X_i|\mu_{-i})$ is the set of contracts that i chooses from a set X_i conditional on a reference set μ_{-i}, which is the set of contracts signed by the other agents on the same side.[8] We expand the domain of the choice function so that $c_i(X|\mu) = c_i(X_i|\mu_{-i})$. Choice function c_i has externalities if there exist $X, \mu, \mu' \subseteq \mathcal{X}$ such that $c_i(X|\mu) \neq c_i(X|\mu')$; otherwise, the choice function exhibits no externalities. Let $r_i(X|\mu) \equiv X_i \setminus c_i(X|\mu)$ be the set of contracts rejected by agent i from X conditional on a reference set μ. Similarly, define $C^\theta(X|\mu) \equiv \cup_{i \in \theta} c_i(X|\mu)$ and $R^\theta(X|\mu) \equiv \cup_{i \in \theta} r_i(X|\mu)$ to be

[8]We could allow choice functions c_i that depend not only on X_i and μ_{-i} but also on μ_i (that is the set of contracts signed by i) with no change in our proofs.

the set of chosen contracts and the set of rejected contracts from set X by side θ conditional on a reference set μ, respectively. Note that for any $X, \mu \subseteq \mathcal{X}$ and side θ, $C^{\theta}(X|\mu)$ and $R^{\theta}(X|\mu)$ form a partition of X since every contract involves exactly one agent from each side of the market and is accepted or rejected by the agent. A **matching problem** is a tuple $(\mathcal{B}, \mathcal{S}, \mathcal{X}, C^{\mathcal{B}}, C^{\mathcal{S}})$.

A **matching** is a set of contracts. We embed any quota constraints, if they exist, in agents' choice behavior. For instance, we model one-to-one matching markets by assuming that each agent chooses at most one contract from any set of contracts. Thus, examples of our setting include standard one-to-one and many-to-one matching problems with and without transfers.[9]

A matching μ is **individually rational** for agent i if $c_i(\mu_i|\mu_{-i}) = \mu_i$. Less formally, conditional on the contracts of other agents on the same side, agent i wants to keep all of their contracts. A buyer i and seller j form a **blocking pair** for matching μ if there exists a contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}|\mu) \cap c_j(\mu \cup \{x\}|\mu)$. In words, a pair can block a matching μ if they both would like to sign a new contract conditional on μ. Matching μ is **stable** if it is individually rational for all agents and there are no blocking pairs. This stability concept is identical to pairwise stability studied in settings without externalities (Gale and Shapley, 1962). As in the standard settings without externalities, stability defined in terms of individual and pairwise blocking is equivalent to group stability when choice rules are substitutable; see Appendix B.

3.1 Properties of Choice Functions

To guarantee the existence of a stable matching, we impose more structure on the choice functions. First, we generalize two standard assumptions studied in the matching literature without externalities to our setting. Then, we introduce a new assumption, which is trivially satisfied when there are no externalities.

The first assumption is a basic rationality axiom that we assume throughout the paper.

Definition 1. Choice function C^{θ} satisfies the **irrelevance of rejected contracts** if for all $X, X', \mu \subseteq \mathcal{X}$, we have

$$C^{\theta}(X'|\mu) \subseteq X \subseteq X' \implies C^{\theta}(X'|\mu) = C^{\theta}(X|\mu).$$

[9] Without affecting any of the results, we could alternatively model one-to-one matching and other matching environments with quota constraints by assuming that only some sets of contracts are feasible matchings. This alternative route is straightforward if agents condition their choice behavior on any sets of contracts rather than on feasible matchings. As is usual in models of matching with contracts, in applications with transfers, we assume that there is a lowest monetary unit.

If choice function C^θ satisfies the irrelevance of rejected contracts, then excluding contracts that are not chosen does not change the chosen set. This is a basic property of choice functions. It has been studied in the matching with contracts literature by Aygün and Sönmez (2013) when there are no externalities. The irrelevance of rejected contracts is satisfied in all our examples.

The second assumption rules out complementarities.

Definition 2. Choice function C^θ satisfies **standard substitutability** if for any $X, X', \mu \subseteq \mathcal{X}$,

$$X' \supseteq X \implies R^\theta(X'|\mu) \supseteq R^\theta(X|\mu).$$

A choice function satisfies standard substitutability if the corresponding rejection function is monotone for a fixed reference set, or equivalently, a contract that is chosen from a set is also chosen from any subset including that contract conditional on the same reference set. When there are no externalities, the choice behavior does not depend on the reference set and this assumption reduces to the condition introduced by Kelso and Crawford (1982) for a matching market with transfers.[10]

By construction, C^θ satisfies standard substitutability (or the irrelevance of rejected contracts) if, and only if, c_i satisfies standard substitutability (or the irrelevance of rejected contracts) for every agent $i \in \theta$. Therefore, we can impose these two conditions on either agents' choice functions or the choice functions for each side of the market.

To introduce our last assumption on choice functions, we need the following concepts. A preorder $\succsim^\theta$ for side θ is **consistent** with the side choice function C^θ if,[11] for any $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X' \supseteq X \text{ and } \mu' \succsim^\theta \mu \implies C^\theta(X'|\mu') \succsim^\theta C^\theta(X|\mu).$$

Furthermore, a preorder $\succsim^\theta$ for side θ is **separable** if, for every $i \in \theta$, there exists a preorder $\succeq_i$ such that

$$\mu' \succsim^\theta \mu \iff \mu'_i \succeq_i \mu_i \ \forall i \in \theta.$$

Separability allows us to restrict the preorder to a subset of agents on side θ. For instance, any separable preorder $\succeq^\theta$ induces preorder $\succeq^{\theta\setminus\{i\}}$ for every agent $i \in \theta$ such that $\mu' \succeq^{\theta\setminus\{i\}} \mu$ if, and only if, $\mu'_j \succeq_j \mu_j$ for every $j \in \theta \setminus \{i\}$.

[10]See also Roth (1984) and Hatfield and Milgrom (2005).

[11]In our context, a **binary relation** $\succeq$ on a domain $\mathcal{A} \subseteq 2^{\mathcal{X}}$ is a set of ordered matchings in $\mathcal{A}$. It is **reflexive** if for any $\mu \in \mathcal{A}$, $\mu \succeq \mu$. It is **transitive**, if $\mu_1 \succeq \mu_2$ and $\mu_2 \succeq \mu_3$ imply $\mu_1 \succeq \mu_3$. A reflexive and transitive binary relation is called a **preorder**.

A consistent and separable preorder $\tilde{\succeq}_1^\theta$ is **minimal** if there does not exist another consistent and separable preorder $\tilde{\succeq}_2^\theta \neq \tilde{\succeq}_1^\theta$ such that for any $\mu, \mu' \subseteq \mathcal{X}$, $\mu \tilde{\succeq}_2^\theta \mu'$ implies $\mu \tilde{\succeq}_1^\theta \mu'$. The following lemma establishes the existence and uniqueness of a minimal preorder that is consistent and separable.

Lemma 1. *There exists a unique minimal preorder that is separable and consistent with the side choice function C^θ.*

Proof. Consider the following preorder $\succeq$: for every $\mu, \mu' \subseteq \mathcal{X}$, $\mu \succeq \mu'$. This preorder is separable and consistent with the choice function C^θ. Hence, there exists at least one such preorder. Now, let us construct a minimal one.

Suppose that $\{\succeq_1^\theta, \succeq_2^\theta, \ldots, \succeq_k^\theta\}$ is the set of all preorders that are separable and consistent with choice function C^θ. Define the following binary relation: $\mu' \succeq^\theta \mu$ if, and only if, $\mu' \succeq_j^\theta \mu$ for every $j = 1, \ldots, k$. The binary relation $\succeq^\theta$ is reflexive and transitive, so it is a preorder. Furthermore, since each $\succeq_j^\theta$ is separable, so is $\succeq^\theta$.

Now we show that $\succeq^\theta$ is consistent with the side choice function C^θ. Let $X' \supseteq X$ and $\mu' \succeq^\theta \mu$. Then $\mu' \succeq_j^\theta \mu$ for every $j = 1, \ldots, k$. By consistency of $\succeq_j^\theta$, we get $C^\theta(X'|\mu') \succeq_j^\theta C^\theta(X|\mu)$ for every $j = 1, \ldots, k$. As a result, $C^\theta(X'|\mu') \succeq^\theta C^\theta(X|\mu)$ by the definition of C^θ. Therefore, $\succeq^\theta$ is also consistent with the choice function C^θ. Since the number of preorders is finite, this argument shows that there exists a unique minimal preorder $\succeq^\theta$, which is separable and consistent with C^θ. ■

We define our last assumption using this minimal preorder $\succeq^\theta$.[12] To simplify exposition, when $\mu' \succeq^\theta \mu$ we say that μ' has a **better market condition** than μ for side θ. The better market condition preorder is a generalization of the revealed-preference **Blair order** (Blair, 1988) when the choice functions do not have externalities: $\mu_i' \succeq_i \mu_i$ if, and only if, $c_i(\mu_i' \cup \mu_i) = \mu_i$. In general, the domain of the better market condition preorder is not the set of all matchings. Therefore, not all pairs of matchings can be compared using the better market condition. This is also true in the special case of the Blair order since it cannot be used to rank all pairs of matchings. For instance, when there are no externalities, $\mu \succeq^\theta \mu$ holds only when μ is a fixed point of C^θ, i.e., when $C^\theta(\mu) = \mu$. Likewise, in our general case, if a matching μ is not in the domain of the better market condition preorder, we do not have $\mu \succeq^\theta \mu$.

[12] Imposing separability on the minimal preorder is not needed for some of the results. For example, Theorem 2, which shows the existence of a stable matching, works when monotone externalities is defined using the minimal consistent preorder. Therefore, the alternative version without separability is stronger. However, the current Theorem 3 is stronger than the alternative one because the constructed choice functions satisfy the stronger version of monotone externalities.

Now we are ready to state our main assumption.

Definition 3. Choice function C^θ satisfies **monotone externalities** if for any $X, \mu, \mu' \subseteq \mathcal{X}$,

$$\mu' \geq^\theta \mu \geq^\theta \varnothing \Longrightarrow R^\theta\left(X|\mu'\right) \supseteq R^\theta\left(X|\mu\right).$$

In words, the choice function of side θ satisfies monotone externalities if any contract rejected from a set X conditional on a reference set μ, that has a better market condition than the empty set, is also rejected from X conditional on a reference set μ' that has a better market condition than μ. Alternatively, a contract chosen from X conditional on μ' is also chosen from X conditional on μ that has a worse market condition than μ' and a better market condition than the empty set.

In monotone externalities, we condition the choice and rejection functions on reference sets; in particular, we impose that μ' has a better market condition than μ. This is a novel property. Importantly, when there are no externalities for side θ, it is satisfied trivially as the rejection function does not depend on the reference set.

Since the minimal preorder $\geq^\theta$ is separable, monotone externalities can equivalently be defined as follows: for any $X, \mu, \mu' \subseteq \mathcal{X}$ and $i \in \theta$,

$$\mu'_{-i} \geq^{\theta\setminus\{i\}} \mu_{-i} \geq^{\theta\setminus\{i\}} \varnothing \Longrightarrow r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i}).$$

When we consider a subset of agents on side θ, say θ', the preorder $\geq^{\theta'}$ is constructed only on sets of contracts associated with these agents. Therefore, monotone externalities for $C^{\theta'}$ checks monotonicity of the rejection function $R^{\theta'}$ conditional on reference sets that has contracts associated with agents in θ' only. Therefore, if choice function C^θ satisfies monotone externalities, then for every $\theta' \subseteq \theta$, choice function $C^{\theta'}$ also satisfies monotone externalities. Furthermore, if there is only one agent on side θ, then C^θ trivially satisfies monotone externalities as in the case of no externalities.

The conjunction of standard substitutability and monotone externalities is equivalent to the following property:

Definition 4. Choice function C^θ satisfies **substitutability** if for any $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X' \supseteq X \text{ and } \mu' \geq^\theta \mu \geq^\theta \varnothing \Longrightarrow R^\theta\left(X'|\mu'\right) \supseteq R^\theta\left(X|\mu\right).$$

If substitutability (or monotone externalities) is satisfied for a separable and consistent preorder, then it is also satisfied for the consistent and separable minimal preorder $\geq^\theta$. This is important because, in our applications, we do not need to construct $\succeq^\theta$. Instead, we just need to verify that substitutability (or monotone externalities) is satisfied for a separable and consistent preorder. The reason is that the minimal preorder $\succeq^\theta$ compares less pairs of reference sets, so substitutability (or monotone externalities) is weaker for the minimal preorder compared to any other consistent and separable preorder.

Remark 1. The standard substitutability condition can be weakened without affecting our results in two different ways. In the first approach, the reference set is restricted to be a set that can be chosen by side θ. More formally, consider the minimal set of matchings $\mathcal{A}^\theta$ that contains the empty set and satisfies $C^\theta(X|\mu) \in \mathcal{A}^\theta$ whenever $X \subseteq \mathcal{X}$ and $\mu \in \mathcal{A}^\theta$. The minimal such domain is $\mathcal{A}^\theta \equiv \bigcup_{t=0,1,\dots} \mathcal{A}_t^\theta$ where $\mathcal{A}_0^\theta \equiv \{\emptyset\}$ and $\mathcal{A}_t^\theta$ for $t \geq 1$ are defined recursively $\mathcal{A}_t^\theta \equiv \{C^\theta(X|\mu) : X \subseteq \mathcal{X}, \mu \in \mathcal{A}_{t-1}^\theta\} \cup \mathcal{A}_{t-1}^\theta$. Since there exists a finite number of contracts, $\mathcal{A}^\theta$ is well-defined; it is the set of all matchings that can be reached from the empty set by applying the choice function C^θ. Standard substitutability can be weakened by imposing it only for reference sets in $\mathcal{A}^\theta$. Furthermore, it can be shown by recursion that $\mu \succeq^\theta \emptyset$ for every matching $\mu \in \mathcal{A}^\theta$, so we can weaken standard substitutability also by requiring that $\mu \succeq^\theta \emptyset$ without referring to $\mathcal{A}^\theta$.

The second approach works only when agents on one side of the market have unit demand using the techniques developed in Hatfield and Kojima (2010), Hatfield and Kominers (2016), and Hatfield, Kominers, and Westkamp (2017) when there are no externalities. These conditions usually proceed by restricting X' and X under which the standard substitutability condition holds. Such conditions can also be studied in our setting when one side of the market can sign at most one contract. Furthermore, a combination of the two approaches can be used when agents on one side of the market have unit demand.

3.2 Illustrative Examples

The following examples illustrate our concepts and notation.

Example 3. Suppose that there are two sellers s_1 and s_2 and two buyers b_1 and b_2. Seller s_1 and buyer b_1 can sign contract x_1 and seller s_1 and buyer b_2 can sign contract x_2. Seller s_2 can sign contract x_3 with buyer b_2 only.[13] The contractual structure is demonstrated in Figure 1.

[13]This example is a special case of Example 1 with the following interpretation. Sellers are firms and buyers are workers. Buyers b_1 and b_2 are married. Buyer b_1 is a woman, so her choice function does not have externalities. Buyer b_2 is a man and the outside option of not working is ranked higher whenever his wife works. In particular, contract x_2 is ranked below the outside option if the wife has a job.

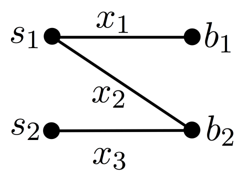

Figure 1: Contractual structure in Example 3.

Seller choice functions do not have externalities. Seller s_1 always chooses one contract, if there exists one, and prefers contract x_2 over x_1 and seller s_2 chooses contract x_3 when it is available. Therefore, seller choice functions satisfy standard substituability. They also satisfy monotone externalities because there are no externalities for sellers.

Buyer b_1 wants to sign contract x_1 regardless of the contracts signed by buyer b_2. Conditional on the empty set, buyer b_2 wants to sign one contract only and prefers contract x_3 to x_2. Conditional on the reference set $\{x_1\}$, b_2 chooses contract x_3, if it is available, and rejects x_2, if it is available. Therefore, the only choice function that has externalities is that of buyer b_2, which is summarized by the following table.

	$\{x_2, x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot \vert \{x_1\})$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$
$c_{b_2}(\cdot \vert \varnothing)$	$\{x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$

Table 1: Choice function of buyer b_2 in Example 3. Columns are indexed by the set of available contracts and rows are indexed by the set of contracts signed by buyer b_1.

First let us construct the better market condition for buyers. Since buyer b_1 chooses contract x_1 whenever it is available, we have $\{x_1\} \succeq_{b_1} \varnothing$. For buyer b_2, using consistency on sets of contracts $\{x_2, x_3\} \supseteq \{x_2\} \supseteq \varnothing$ with the empty set as a reference set, we get $\{x_3\} \succeq_{b_2} \{x_2\} \succeq_{b_2} \varnothing$. In addition, since $\{x_1\} \succeq_{b_1} \varnothing$, $c_{b_2}(\{x_2\}|\{x_1\}) = \varnothing$, and $c_{b_2}(\{x_2\}|\varnothing) = \{x_2\}$, we get $\varnothing \succeq_{b_2} \{x_2\}$. The better market condition for buyers $\succeq^{\mathcal{B}}$ is then defined as $\mu' \succeq^{\mathcal{B}} \mu \Leftrightarrow \mu'_{b_i} \succeq_{b_i} \mu_{b_i}$ for every $i \in \{1, 2\}$.

It is easy to check that standard substitutability is also satisfied for the buyers. To check monotone externalities, note that choice function of buyer b_1 does not have externalities, so it does not depend on the reference set and the choice function of buyer b_2 rejects more contracts when it is conditional on the reference set $\{x_1\}$ rather than the reference set $\varnothing$, where $\{x_1\} \succeq^{\mathcal{B}} \varnothing$. ■

Example 4. We modify Example 3 by changing the choice function of buyer b_2. Buyer b_2 chooses all available contracts conditional on the reference set $\{x_1\}$. Furthermore, conditional on the empty set, she signs contract x_3, if it is available, and rejects x_2, if it is available. Choice function of buyer b_2 is summarized by the following table.

	$\{x_2, x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_2, x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\varnothing)$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$

Table 2: Choice function of buyer b_2 in Example 4. Columns are indexed by the set of available contracts and rows are indexed by the set of contracts signed by buyer b_1.

As in the previous example, it is easy to check that standard substitutability is satisfied for buyers. However, monotone externalities fails. To see this, note that for any consistent preorder we need $\{x_1\} \geq^{\mathcal{B}} \varnothing$. But conditional on $\{x_1\}$, buyer b_2 accepts more contracts than conditional on the empty set.

While our general results imply that there exist stable matchings in the earlier examples, it is easy to see that there is no stable matching Example 4: Matchings $\varnothing$ and $\{x_3\}$ are blocked by seller s_1 and buyer b_1 via contract x_1. Matchings $\{x_1\}$ and $\{x_1, x_2\}$ are blocked by seller s_2 and buyer b_2 via contract x_3. Matchings $\{x_2\}$ and $\{x_2, x_3\}$ are not individually rational for buyer b_2. Matching $\{x_1, x_3\}$ is blocked by seller s_1 and buyer b_2 via contract x_2. The last remaining matching, $\mathcal{X}$, is not individually rational for seller s_1. ■

3.3 Motivational Examples Revisited

Now, we argue that choice functions introduced in Section 3.2 satisfy substitutability.

Example 1 revisited: Worker choice functions satisfy substitutability for preorder $\geq^{\theta}$ such that $\mu' \geq^{\theta} \mu$ when each married woman gets a weakly more preferred job in μ' compared to μ. This preorder is separable. Furthermore, it is consistent because as there are more contracts available, married women get weakly more preferred jobs since their choice functions do not have externalities. Substitutability is satisfied because a married man becomes weakly more selective whenever his wife gets a more preferred job, so he rejects more contracts conditional on μ' compared to μ whenever $\mu' \geq^{\theta} \mu$. ■

Example 2 revisited: College choice functions satisfy substitutability if we define the preorder $\geq^{\theta}$ so that $\mu' \geq^{\theta} \mu$ if, and only if, $\max_{j\in\mu'(i)} \lambda(i, j)$ is weakly greater than $\max_{j\in\mu(i)} \lambda(i, j)$ for all colleges i.[14] By construction this preorder is separable. Furthermore, it is consistent

[14]When $\mu(i)$ is empty, we set the maximum equal to $-\infty$.

with the choice functions: when more academics are available then the maximum quality of the academics a college hires goes up (whether or not the benchmark quality of academics increases). The substitutability condition is then satisfied: when more academics are available and when the benchmark quality of academics increases, each college continues to reject the academics it previously rejected. ■

3.4 Substitutable Choice Functions: A General Characterization

In this subsection, we study the structure of choice functions that satisfy substitutability. In formulating our characterization result, we use the standard matching concept of truncation (see Roth and Rothblum (1999)). Linear order $\succ'$ over $\mathcal{X}_i \cup \{\varnothing\}$ is a **truncation** of linear order $\succ$ over $\mathcal{X}_i \cup \{\varnothing\}$ if, for all $x, y \in \mathcal{X}_i$ the following two implications hold true:

- $x \succ' \varnothing$ implies $x \succ \varnothing$, and
- $x \succ' y \succ' \varnothing$ implies $x \succ y \succ \varnothing$.

The first bullet states that any contract ranked above the empty set by the linear order $\succ'$ is also ranked above the empty set by the linear order $\succ$. The second bullet point states that the relative ranking of any two contracts preferred to the empty set in the linear order $\succ'$ is preserved in the linear order $\succ$.

The next result characterizes choice functions satisfying our substitutability condition.

Theorem 1. ***(Characterization of Substitutability)*** *Choice function C^θ satisfies substitutability if, and only if, for every agent $i \in \theta$ there is a nonempty set $\mathcal{J}$ and linear orders $\succ_j^{\mu_{-i}}$ over $\mathcal{X}_i \cup \{\varnothing\}$ indexed by $j \in \mathcal{J}$ and matching μ_{-i} that does not include i's contracts such that if $\mu'_{-i} \succeq^\theta \mu_{-i} \succeq^\theta \varnothing$ then for any $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Furthermore, for any $X, \mu \subseteq \mathcal{X}$,*

$$c_i(X_i|\mu_{-i}) = \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\},$$

where $x_j^{\mu_{-i}}$ is the maximum element of $X_i \cup \{\varnothing\}$ in order $\succ_j^{\mu_{-i}}$.

This result is inspired by the Aizerman and Malishevski (1981) decomposition result for substitutable functions when there are no externalities. It states that the choice function can be constructed from a set of linear orders over individual contracts such that the choice from a set conditional on a reference set is the union of the most-preferred contracts with respect to these linear orders. In this representation, the linear orders depend on the reference set and as the reference set gets better with respect to the better market condition the linear orders are truncated.

4 Stable Matchings

As in classical matching theory, a key step in proving the existence of a stable matching is an algorithm akin to the deferred acceptance algorithm.

Our generalization of the deferred acceptance algorithm has two phases. First, we construct an auxiliary matching μ^* such that $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \leq^{\mathcal{S}} \mu^*$. Then, we use μ^* to construct a stable matching in a way resembling the classic deferred acceptance algorithm of Gale and Shapley (1962) and, particularly, its extension by Hatfield and Milgrom (2005): we run the algorithm in rounds, $t = 1, 2, \ldots$. In any round $t \geq 1$, we denote by $A^s(t)$ and $A^b(t)$ the set of contracts that are available to the sellers and buyers, respectively. Therefore, the set of contracts held at the beginning of each round is $A^s(t) \cap A^b(t)$. We also track the reference sets for each side: $\mu^s(t)$ is the seller reference set and $\mu^b(t)$ is the buyer reference set.[15]

Phase 1: Construction of an auxiliary matching μ^* such that $\mu^* \geq^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$. Set $\mu_0 \equiv \emptyset$ and define recursively $\mu_k \equiv C^{\mathcal{S}}(\mathcal{X}|\mu_{k-1})$ for every $k \geq 1$. Since the number of contracts is finite, so is the number of sets of contracts. Therefore, there exist m and $n \leq m$ such that $\mu_{m+1} = \mu_n$. Let $m^* = \min\{m | \exists n \leq m \text{ s.t. } \mu_{m+1} = \mu_n\}$. Let $\mu^* \equiv \mu_{m^*}$. In the proof of Theorem 2, we establish that $\mu^* \geq^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$.

Phase 2: Construction of a stable matching. Set $A^s(1) \equiv \mathcal{X}$ (all contracts are available to the sellers), $A^b(1) \equiv \emptyset$ (no contracts are available to the buyers), and the reference sets are $\mu^s(1) \equiv \mu^*$, and $\mu^b(1) \equiv \emptyset$. In each round $t = 1, 2, \ldots$, we update these sets and matchings as

[15] The tracking of reference sets has no counterpart in earlier formulations of the deferred acceptance algorithms of, among many others, Gale and Shapley (1962), Roth (1984), Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Echenique and Yenmez (2007), Ostrovsky (2008), Hatfield and Kojima (2010), and Bando (2014). In these papers, there is no need to track reference sets and the deferred acceptance algorithm terminates when there are no more rejections and no new offers. However, in our setting, the lack of rejections and new offers is not sufficient to stop the algorithm and we need to run it until the reference sets converge. We run the algorithm in a symmetric way: in each round agents on both sides respond to the offers and rejections from the previous round. This is formally different from the standard approach where agents on the proposing side respond to rejections from the earlier round but the agents on the accepting side respond to offers in the current round. This difference is not substantive: we could run the deferred acceptance algorithm in the latter manner with straightforward adjustments.

follows:

$$
\begin{aligned}
A^s(t+1) &\equiv \mathcal{X} \setminus R^{\mathcal{B}}(A^b(t)|\mu^b(t)), \\
A^b(t+1) &\equiv \mathcal{X} \setminus R^{\mathcal{S}}(A^s(t)|\mu^s(t)), \\
\mu^s(t+1) &\equiv C^{\mathcal{S}}(A^s(t)|\mu^s(t)), \text{ and} \\
\mu^b(t+1) &\equiv C^{\mathcal{B}}(A^b(t)|\mu^b(t)).
\end{aligned}
$$

Thus, the buyers reject some of the contracts available in $A^b(t)$ conditional on the reference set $\mu^b(t)$ and the set of contracts not rejected by the buyers is available to the sellers in the next round, i.e., $A^s(t+1) = \mathcal{X} \setminus R^{\mathcal{B}}(A^b(t)|\mu^b(t))$. Likewise, the sellers reject some contracts available in $A^s(t)$ conditional on the reference set $\mu^s(t)$ and the set of contracts that are not rejected by the sellers is available to the buyers in the next round, i.e., $A^b(t+1) = \mathcal{X} \setminus R^{\mathcal{S}}(A^s(t)|\mu^s(t))$. We also update the reference sets: at the next round, the sellers' reference set is the set of contracts that sellers choose from $A^s(t)$ conditional on $\mu^s(t)$ and likewise for the buyers. We continue updating these sets until round T such that $A^s(T+1) = A^s(T)$, $A^b(T+1) = A^b(T)$, $\mu^s(T+1) = \mu^s(T)$, and $\mu^b(T+1) = \mu^b(T)$. The outcome of the algorithm is then $A^s(T) \cap A^b(T)$.

This is the seller-proposing version of the deferred-acceptance algorithm. The buyer-proposing version can be defined analogously. The main result of this section establishes that the algorithm terminates at some round despite the presence of externalities and, furthermore, it produces a stable matching.

Theorem 2. *(**Sufficiency**) Suppose that the choice functions satisfy substitutability. Then, the algorithm terminates at some finite round T, its outcome $A^s(T) \cap A^b(T)$ is stable, and*

$$
\mu^s(T) = \mu^b(T) = A^s(T) \cap A^b(T).
$$

This result implies that a stable matching exists in environments satisfying substitutability like the examples of Sections 2 and 7. Its proof relies on monotonicity of a function that we define in Section 4.2 but we need to address two complications. First, the second phase of our deferred acceptance procedure is monotonic only in some circumstances; it is the role of the first phase to guarantee monotonicity of the second phase. Second, because we work with preorders rather than partial orders and the domain of the function that we analyze is not a lattice, we cannot use Tarski's fixed-point theorem, which is routinely used in the matching literature (e.g., see Adachi, 2000). Instead, we find that the iterative application in phase two leads us to a set of matchings that are equivalent in the preorder. The relation between sets $A^s(T)$, $A^b(T)$, and reference sets $\mu^s(T)$ and $\mu^b(T)$ allows us to then conclude that $A^s(T) \cap A^b(T)$ is a stable matching. We provide the details of our proof in Appendix C.

Next we provide a result which shows that monotone externalities is necessary for the existence of a stable matching in a "maximal domain" sense when standard substitutability is satisfied.

Theorem 3. *(**Necessity**) Suppose that there exists an agent i on side θ such that c_i has externalities and satisfies standard substitutability. Then, there exist substitutable choice functions for the remaining agents on side θ and substitutable choice functions without externalities for agents on side $-\theta$ such that no stable matching exists.*

Notice that in this theorem the choice function c_i is fixed while choice functions of other agents are constructed. For this reason, we do not make any assumptions on whether monotone externalities is satisfied or not; however, for the constructed choice profile monotone externalities and substitutability fail for side θ, and standard substitutability is satisfied for all agents.

In the proof, we construct choice functions of agents other than i that satisfy the properties stated above. Each agent on side $-\theta$ has a choice function without externalities such that any contract specified in a set is chosen whenever it is available. Since these choice functions do not exhibit externalities, monotone externalities is trivially satisfied. Furthermore, these choice functions also satisfy standard substitutability. On the other hand, each remaining agent on side θ has two different sets of contracts that they like. They choose all the contracts available from one of these two sets depending on whether i has signed a particular contract or not. Therefore, these choice functions satisfy standard substitutability. Furthermore, since the contract of i can never be chosen by the remaining agents on side θ, $C^{\theta\setminus\{i\}}$ satisfies monotone externalities. We explain the details of the proof in Appendix C.

When there are no externalities, Hatfield and Kominers (2017) show that standard substitutability is a necessary condition for the existence of a stable matching in many-to-many matching markets. In contrast, we assume standard substitutability and show that monotone externalities is a necessary condition for the existence of a stable matching in many-to-many matching markets when there are externalities.

In the reminder of this section, we discuss the similarities and differences of our algorithm with the standard deferred acceptance algorithm, provide an illustration of how it runs, and establish two auxiliary properties of the transformation iteratively performed in the second phase.

4.1 Illustration of the Deferred Acceptance Algorithm

Like the standard deferred acceptance algorithm, in each round of phase 2, substitutability and monotone externalities imply that $A^s(t+1) \subseteq A^s(t)$ and $A^b(t+1) \supseteq A^b(t)$, i.e., the sellers make more offers to the buyers while the buyers reject more contracts with each passing round (Lemma 2). As a consequence, the sellers' reference set gets worse and the buyers' reference set gets better. Hence, both of these two sets converge at some round t; however, the algorithm does not necessarily terminate when $A^s(t+1) = A^s(t)$ and $A^b(t+1) = A^b(t)$. Indeed, because of externalities, the set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is not necessarily stable. Instead, the algorithm converges only when $A^s(t+1) = A^s(t)$, $A^b(t+1) = A^b(t)$, $\mu^s(t+1) = \mu^s(t)$, and $\mu^b(t+1) = \mu^b(t)$. The set of contracts held, $A^s(t) \cap A^b(t)$, at such a round is stable.

The next example, which is based on Example 3, illustrates this point and shows the steps of the algorithm. It also demonstrates that our algorithm can be viewed as an ascending auction in the presence of externalities.

Example 3 revisited. In the first phase, we start with $\mu_0 = \emptyset$. Then, $\mu_1 = C^{\mathcal{S}}(\mathcal{X}|\mu_0) = \{x_2, x_3\}$, and $\mu_2 = C^{\mathcal{S}}(\mathcal{X}|\mu_1) = \{x_2, x_3\}$. Since $\mu_1 = \mu_2$, we can set $\mu^* = \{x_2, x_3\}$.

In the first round of the second phase, all contracts are available to the sellers, so they choose $\{x_2, x_3\}$. However, no contract is available to the buyers, so they choose the empty set. Therefore, in the second round, the seller reference set is $\{x_2, x_3\}$ and the buyer reference set is the empty set. In addition, the set of contracts available to the buyers is the set of contracts not rejected by the sellers at the first round, which is $\{x_2, x_3\}$.

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\|\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\|\mu^b(t))$
$t=1$	$\mathcal{X}$	$\emptyset$	$\{x_2, x_3\}$	$\emptyset$	$\{x_2, x_3\}$	$\emptyset$
$t=2$	$\mathcal{X}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\emptyset$	$\{x_2, x_3\}$	$\{x_3\}$
$t=3$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_3\}$	$\{x_1, x_3\}$	$\{x_3\}$
$t=4$	$\{x_1, x_3\}$	$\mathcal{X}$	$\{x_1, x_3\}$	$\{x_3\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$
$t=5$	$\{x_1, x_3\}$	$\mathcal{X}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$
$t=6$	$\{x_1, x_3\}$	$\mathcal{X}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$		

Table 3: Rounds of the Deferred Acceptance Algorithm in Example 3.

The algorithm continues to proceed in this way. Table 3 shows all the rounds. Notice that between the fourth and fifth rounds the sets of contracts available to the buyers and sellers are the same, i.e., $A^b(4) = A^b(5)$ and $A^s(4) = A^s(5)$. In the standard deferred acceptance algorithm, we could stop the algorithm here. In our setting, the deferred acceptance does not converge yet because the reference sets for the buyers are different at these two rounds. The algorithm eventually converges at the sixth round and produces the matching $A^s(6) \cap A^b(6) = \{x_1, x_3\}$, which is stable: It is individually rational for all agents. There is only one potential blocking pair (s_1, b_2) via contract x_2 but they do not block this matching because $x_2 \notin c_{b_2}(\{x_2, x_3\}|\{x_1\})$.

Note that the set of contracts available to the sellers, $A^s(t)$, is shrinking and the set of contracts available to the buyers, $A^b(t)$, is expanding as the algorithm proceeds. Likewise, the seller reference set $\mu^s(t)$ is getting worse for the sellers and the buyer reference set $\mu^b(t)$ is getting better for the buyers. ■

When choice functions satisfy standard substitutability, DA produces a stable matching *if* it converges (see Theorem 4 and Lemma 3 below). However, when monotone externalities fails, it does not have to converge and a stable matching need not exist. We show these two claims with the following example.

Example 4 revisited. The first phase works as in the previous example since seller choice functions satisfy substitutability. The algorithm starts diverging after round five of the second phase because conditional on the reference set $\mu^b(5) = \{x_1, x_3\}$, the buyers choose all contracts. Table 4 shows the first nine rounds of DA. At round nine, we get the same sets of contracts

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\|\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\|\mu^b(t))$
$t=1$	$\mathcal{X}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$
$t=2$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\{x_3\}$
$t=3$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_3\}$
$t=4$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=5$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\mathcal{X}$
$t=6$	$\mathcal{X}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\mathcal{X}$
$t=7$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$
$t=8$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$
$t=9$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		

Table 4: Rounds of the Deferred Acceptance Algorithm in Example 4.

available to the buyers and sellers and the same reference sets as in round three. Therefore, the algorithm does not converge. This outcome is not surprising because we Section 3 showed that there is no stable matching in this example. ■

4.2 A Characterization of Stable Matchings via Fixed Points of a Monotone Function

Let us introduce some notation for the proofs of Theorem 2 and the subsequent results. Each iteration in the second phase of our algorithm can be described as the following transformation function

$$f\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X} \backslash R^{\mathcal{B}}(A^b|\mu^b), \mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s), C^{\mathcal{S}}\left(A^s|\mu^s\right), C^{\mathcal{B}}(A^b|\mu^b)\right),$$

where f is a function from $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ into itself.

Function f has two important properties, monotonicity and stability of its fixed points, that are captured in the following auxiliary results.

Lemma 2. *Suppose that the choice functions satisfy standard substitutability and monotone externalities. Then function f is monotone increasing with respect to the preorder $\sqsubseteq$ defined as follows:*

$$(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b) \Longleftrightarrow A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succeq^{\mathcal{B}} \tilde{\mu}^b.$$

The fixed points of f satisfy the following properties even when the choice functions do not satisfy substitutability or monotone externalities.

Lemma 3. *Let $\left(A^s, A^b, \mu^s, \mu^b\right)$ be a fixed point of function f. Then $A^s \cup A^b = \mathcal{X}$ and*

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}(A^s|\mu^s).$$

When choice functions satisfy substitutability, a matching is stable if, and only if, it can be supported as a fixed point of f.

Theorem 4. ***(Characterization of Stability)*** *Suppose that the choice functions satisfy standard substitutability. Then a matching μ is stable if, and only if, there exist sets of contracts $A^s, A^b \subseteq \mathcal{X}$ such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of function f.*

The proofs of these three results are provided in Appendix C.

5 Pareto Efficiency and Side-Optimal Stable Matchings

Two key normative properties in the standard theory of stable matchings is Pareto efficiency of stable matchings and the existence of side-optimal stable matchings. Pareto efficiency extends to our setting as follows:

Theorem 5. *(**Pareto Efficiency**) Suppose that the choice functions satisfy substitutability. If matching μ is stable then it is Pareto efficient in the following sense: there is no other matching $\nu \neq \mu$ such that $\nu_i = c_i(\nu \cup \mu|\mu)$ for every agent i.*

The argument resembles a similar argument in the no-externalities case. We prove a stronger result in Appendix B (Proposition 1).

The counterpart of the side-optimal stable matchings in the setting with externalities is more subtle and it is given by the following result. Before stating this result, we define the following concepts.

Definition 5. A stable matching μ is θ-**optimal** if $\mu \geq^{\theta} \mu'$ for every stable matching μ', it is θ-**pessimal** if $\mu \leq^{\theta} \mu'$ for every stable matching μ'.

In the standard stable matching theory without externalities, side optimality is measured with respect to the revealed preference of agents on this side (e.g., Blair, 1988). This standard result is subsumed.

Theorem 6. *(**Side Optimality**) Suppose that the choice functions satisfy substitutability and, in addition, for side θ there exists a matching $\bar{\mu}^{\theta}$ such that for any matching μ, $\bar{\mu}^{\theta} \geq^{\theta} \mu$. Then, the θ-proposing deferred-acceptance algorithm when the reference set for side θ is $\bar{\mu}^{\theta}$ produces a θ-optimal stable matching, which is also a $-\theta$-pessimal stable matching.*

Proof. Without loss of generality assume that $\theta = \mathcal{S}$. For any $(A^s, A^b, \mu^s, \mu^b) \in 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ we have $(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) \sqsupseteq (A^s, A^b, \mu^s, \mu^b)$. Therefore, $(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) \sqsupseteq f(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing)$. By Lemma 2, function f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^{k-1}(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) \sqsupseteq f^k(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing)$ for every $k \geq 1$. Since $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ is a finite set, this sequence converges at some point as in the proof of Theorem 2, so there exists k such that $f^{k-1}(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) = f^k(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing)$. Therefore, $f^{k-1}(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing)$ is a fixed point of f. By Lemma 3 there is $\left(\hat{A}^s, \hat{A}^b, \hat{\mu}, \hat{\mu}\right)$ that is equal to $f^{k-1}(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing)$. Theorem 4 tells us that $\hat{\mu}$ is a stable matching, which is the outcome of the seller-proposing deferred-acceptance algorithm.

We next show that $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. Let μ be any stable matching. By Theorem 4, there exist A^s and A^b such that (A^s, A^b, μ, μ) is a fixed point of f. Since $(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) \sqsupseteq (A^s, A^b, \mu, \mu)$ and f is monotonic increasing, f can be applied repeatedly while preserving the order. Therefore, $f^k(\mathcal{X}, \varnothing, \bar{\mu}^s, \varnothing) \sqsupseteq f^k(A^s, A^b, \mu, \mu)$ for every k, which implies $\left(\hat{A}^s, \hat{A}^b, \hat{\mu}, \hat{\mu}\right) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $\hat{\mu} \succeq^{\mathcal{S}} \mu$ and $\hat{\mu} \preceq^{\mathcal{B}} \mu$, so $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. ■

The assumption that there exists a matching $\bar{\mu}^\theta$ such that for any matching μ, $\bar{\mu}^\theta \succeq^\theta \mu$ plays a crucial role in the proof of Theorem 6. It is not innocuous but it is satisfied in all the examples of Section 2 and Appendix D. In the absence of externalities, this assumption is automatically satisfied because $\succeq^\theta$ is the Blair order. Indeed, for this special case, we can take $\bar{\mu}^\theta$ to be $C^\theta(\mathcal{X})$. Then for any matching μ, $\mathcal{X} \supseteq \bar{\mu}^\theta \cup \mu \supseteq C^\theta(\mathcal{X}) = \bar{\mu}^\theta$ and the irrelevance of rejected contracts yield $C^\theta(\bar{\mu}^\theta \cup \mu) = C^\theta(\mathcal{X}) = \bar{\mu}^\theta$. This implies $\bar{\mu}^\theta \succeq^\theta \mu$. Thus, Theorem 6 subsumes the standard insight that, in the absence of externalities, there exists a θ-optimal stable matching with respect to the Blair order if choice functions satisfy substitutability. This matching is also $(-\theta)$-pessimal.

Furthermore, our assumption on $\bar{\mu}^\theta$ is equivalent to the following: for any two matchings μ and μ', there exists another matching $\tilde{\mu}$ such that $\tilde{\mu} \succeq^\theta \mu$ and $\tilde{\mu} \succeq^\theta \mu'$. In fact, in light of our analysis of DA above, it is enough to impose this assumption on matchings μ such that $C^\theta(\mathcal{X}|\mu) \preceq^\theta \mu$.

Before we end this section, we provide an example which shows the assumption that there exists a side-optimal matching is necessary for Theorem 6.

Example 5. Suppose that there are two buyers b_1, b_2 and one seller, s_1. There is only one contract associated with every seller-buyer pair. Let the contract between b_1 and s_1 be x_1 and the contract between b_2 and s_1 be x_2. Since there is only one seller, there are no externalities for the seller side.

Choice functions are as follows: Seller s_1 chooses all contracts available. Buyer b_1 chooses x_1 conditional on the reference set $\{x_2\}$ and rejects x_1 conditional on the empty set. Buyer b_2 chooses x_2 conditional on the reference set $\{x_1\}$ and rejects x_2 conditional on the empty set. That is each buyer chooses their contract only if the other buyer has the other contract.

Choice function of the seller satisfies substitutability. For buyers, consider the preorder $\succeq^{\mathcal{B}}$ with the domain $\{\varnothing\}$ such that $\varnothing \succeq^{\mathcal{B}} \varnothing$. Thus this preorder does not compare any other pairs of matchings.[16] This preorder is separable and consistent because conditional on the empty set both buyers do not choose any contract. In addition, the buyer-side choice function satisfies

[16] In general, we allow the domain of the preorder to be smaller than the set of all matchings, which is the case in this example.

substitutability because the buyer-side rejection function is monotone conditional on the empty set.

There exists no buyer-optimal stable matching in this example because both the empty set and $\{x_1, x_2\}$ are stable matchings which cannot be compared by the preorder $\geq^{\mathcal{B}}$. This is compatible with Theorem 6 because there exists no buyer-optimal matching $\bar{\mu}^{\mathcal{B}}$ such that $\bar{\mu}^{\mathcal{B}} \geq^{\mathcal{B}} \mu$ for all matchings μ, which is the additional assumption needed for the existence of a side-optimal stable matching. ∎

6 Comparative Statics

How do stable matchings change when agents' choice functions stop (or begin) exhibiting externalities? We answer this question while controlling for the agents' propensity to reject contracts.

Definition 6. Choice function C^θ is an **expansion** of choice function $\hat{C}^\theta$ if $C^\theta(X|\mu) \supseteq \hat{C}^\theta(X|\mu)$ for any $\mu, X \subseteq \mathcal{X}$; we then also say that $\hat{C}^\theta$ is a **contraction** of C^θ.

In words, when choice function C^θ is an expansion of choice function $\hat{C}^\theta$, it admits weakly more contracts (in the superset sense) than $\hat{C}^\theta$ for any set of available contracts and reference set. Likewise, a contraction of a choice function selects weakly less contracts for any set of contracts and reference set. A natural instance of contraction is when contracts are substitutes under both $\hat{C}^\theta$ and C^θ and contracts are *closer substitutes* under $\hat{C}^\theta$ than under C^θ: the strength of substitutability among two contracts being measured by whether an agent is willing to choose both of them or not. For instance, in Example 2, when a college has larger k, which is the share of other colleges it benchmarks itself against, it becomes more reluctant to hire more than one academic making academics closer substitutes for this college.[17]

Controlling for the agents' propensity to reject contracts allows us to establish unambiguous comparative statics: removing externalities while contracting choice for one side of the market benefits this side and harms the other side.

[17] At the same time we developed our analysis, related issues were also studied by Echenique and Yenmez (2015) and Chambers and Yenmez (2017); they introduced the terminology of choice function C^θ being an expansion of choice function $\hat{C}^\theta$ while we originally used the terminology of C^θ exhibiting weaker substitutes than $\hat{C}^\theta$; cf. also Kamada and Kojima (2019). However, even when specialized to the setting without externalities, the result of this section does not have a direct counterpart in the literature. Furthermore, we can weaken the comparison by imposing our condition only when $C^\theta(X|\mu) = \mu$; the weaker assumptions suffice because in the proof we apply this condition to $C^{\mathcal{S}}$ and $\hat{C}^{\mathcal{S}}$ only when $C^{\mathcal{S}}(A^s|\mu) = \mu$.

Theorem 7. *(**Comparative Statics**) Suppose that the choice functions $C^{\mathcal{B}}$, $C^{\mathcal{S}}$, and $C^{*\mathcal{S}}$ satisfy substitutability, $C^{\mathcal{S}}$ does not exhibit externalities and it is a contraction of $C^{*\mathcal{S}}$. Then, for any $\left(C^{\mathcal{B}}, C^{*\mathcal{S}}\right)$-stable matching μ^* there exists a $\left(C^{\mathcal{B}}, C^{\mathcal{S}}\right)$-stable matching μ such that*

$$\mu \geq^{\mathcal{S}} \mu^* \ \ \text{and} \ \ \mu^* \geq^{\mathcal{B}} \mu.$$

In particular, all sellers weakly $C^{\mathcal{S}}$ Blair prefer μ to μ^*, and all buyers whose choice functions do not exhibit externalities weakly Blair prefer μ^* to μ.

We prove the displayed comparisons in Appendix D. The corollary on Blair order follows because we can modify the preorder $\geq^{\mathcal{B}}$ and $\geq^{\mathcal{S}}$ so that they reflect the Blair orders for agents without externalities while maintaining the consistency and separability of the preorders. The modification is straightforward: we replace for all no-externalities agents their individual preorders $\geq_i$ with their Blair order $\gtrsim_i$. As long as the choice behavior is substitutable such a modification of the preorder maintains its consistency with the choice behavior.

While in our discussion we focus on externalities, Theorem 7 is also new for the standard case without externalities. In this case, an example of the contraction of $C^{*\mathcal{S}}$ to $C^{\mathcal{S}}$ is the retirement of a seller; indeed, the retirement of a seller can be modeled as the change in this seller's choice function to rejecting all contracts. In the case without externalities, an analogue of this special case of our result was first established by Crawford (1991) who showed—under the substitutes assumption—that when an agent retires from a market then for each stable matching before the retirement there is a stable matching after the retirement in which all non-retiring agents on the same side weakly gain and all agents on the opposite side weakly lose, just as in our more general result.[18]

7 Conclusion

In this paper, we have studied a two-sided matching problem with externalities where each agent's choice depends on other agents' contracts. For such settings, we have developed the theory of stable matchings by introducing a new substitutability condition when externalities are present. More explicitly, we have studied the existence of stable matchings, Pareto efficiency of stable matchings, side-optimal stable matchings, the deferred acceptance algorithm,

[18]The substitute assumption plays a key role in Crawford's result and hence in ours; see Pycia (2012) for an analysis.

and the rural hospitals theorem (which is in Appendix A). Unlike the previous matching literature, we have not relied on fixed point theorems; instead, we have used elementary techniques to overcome the difficulties associated with externalities.

We believe that our notion of substitutability will be useful to study other important questions in matching markets with externalities. For example, the relation between pairwise stability, group stability, core, and other stability concepts has been an important question in classical matching theory at least since Blair (1988). We analyze the relation between pairwise and group stability in Appendix B, but many related questions remain open. The strategy-proofness of deferred acceptance algorithm (for the proposing side) has been another important question extensively studied since Dubins and Freedman (1981). We think that a deferred acceptance procedure remains strategy-proof in our setting provided we impose the law of aggregate demand à la Hatfield and Milgrom (2005); we leave an exploration of this question for future work. Furthermore, even though we have studied two-sided markets, we think that our techniques are applicable to more general markets such as the supply chain networks of Ostrovsky (2008) where externalities may naturally appear. Subsequent to our work, some of these questions and related ones have been investigated in Rostek and Yoder (2019a) and Rostek and Yoder (2019b).

References

ADACHI, H. (2000): "On a Characterization of Stable Matchings," *Economic Letters*, 68, 43–49. 15, 4

AIZERMAN, M. A., AND A. V. MALISHEVSKI (1981): "General theory of best variants choice: Some aspects," *Automatic Control, IEEE Transactions on*, 26(5), 1030–1040. 1, 3.4, 7

ALI, O. (2016): "Stability in a Many-to-one Matching Model with Positive Externalities among Colleagues," Working Paper. 6

ALKAN, A., AND D. GALE (2003): "Stable Schedule Matching under Revealed Preference," *Journal of Economic Theory*, 112(2), 289 – 306. 7

ASHLAGI, I., M. BRAVERMAN, AND A. HASSIDIM (2014): "Stability in Large Matching Markets with Complementarities," *Operations Research*, 62(4), 713–732. 1

Ashlagi, I., and P. Shi (2014): "Improving Community Cohesion in School Choice via Correlated-Lottery Implementation," *Operations Research*, 62(6), 1247–1264. 1

Aygün, O., and T. Sönmez (2013): "Matching with Contracts: Comment," *American Economic Review*, 103(5), 2050–2051. 3.1, 2

Baccara, M., A. Imrohoroglu, A. J. Wilson, and L. Yariv (2012): "A Field Study on Matching with Network Externalities," *American Economic Review*, 102(5), 1773–1804. 6

Bando, K. (2012): "Many-to-One Matching Markets with Externalities among Firms," *Journal of Mathematical Economics*, 48(1), 14–20. 1

——— (2014): "A Modified Deferred Acceptance Algorithm for Many-to-One Matching Markets with Externalities among Firms," *Journal of Mathematical Economics*, 52(0), 173 – 181. 1, 15

Blair, C. (1988): "The Lattice Structure of the Set of Stable Matchings with Multiple Partners," *Mathematics of Operations Research*, 13(4), 619–628. 1, 3.1, 5, 7

Bodine-Baron, E., C. Lee, A. Chong, B. Hassibi, and A. Wierman (2011): "Peer Effects and Stability in Matching Markets," in *Algorithmic Game Theory*, ed. by G. Persiano, vol. 6982 of *Lecture Notes in Computer Science*, pp. 117–129. Springer Berlin Heidelberg. 5

Chade, H., and J. Eeckhout (2019): "Competing Teams," *The Review of Economic Studies*. 1

Chambers, C. P., and M. B. Yenmez (2017): "Choice and Matching," *American Economic Journal: Microeconomics*, 9(3), 126–47. 3, 17, 19

Chen, B. (2013): "Assignment Games with Externalities and Matching-based Cournot Competition," Discussion papers, Bonn Graduate School of Economics. 5

Chowdhury, P. R. (2004): "Marriage Markets with Externalities," Indian Statistical Institute, Planning Unit, New Delhi Discussion Papers 04-11, Indian Statistical Institute, New Delhi, India. 5

Crawford, V. P. (1991): "Comparative Statics in Matching Markets," *Journal of Economic Theory*, 54, 389–400. 4, 6, 18

DUBINS, L. E., AND D. A. FREEDMAN (1981): "Machiavelli and the Gale-Shapley algorithm," *The American Mathematical Monthly*, 88(7), 485–494. 7

DUR, U. M., AND T. WISEMAN (2019): "School choice with neighbors," *Journal of Mathematical Economics*. 1

DUTTA, B., AND J. MASSÓ (1997): "Stability of Matchings When Individuals Have Preferences over Colleagues," *Journal of Economic Theory*, 75(2), 464–475. 1

ECHENIQUE, F., AND J. OVIEDO (2004): "Core Many-to-One Matchings by Fixed Point Methods," *Journal of Economic Theory*, 115(2), 358–376. 15

——— (2006): "A Theory of Stability in Many-to-Many Matching Markets," *Theoretical Economics*, 1(2), 233–273. 15, 7

ECHENIQUE, F., AND M. B. YENMEZ (2007): "A Solution to Matching with Preferences over Colleagues," *Games and Economic Behavior*, 59(1), 46–71. 1, 15

——— (2015): "How to Control Controlled School Choice," *American Economic Review*, 105(8), 2679–2694. 4, 17

ELLISON, G. (2005): "A Model of Add-On Pricing," *The Quarterly Journal of Economics*, 120(2), 585–637. 25

ERIKSSON, K., F. JANSSON, AND T. VETANDER (2011): "The Assignment Game with Negative Externalities and Bounded Rationality," *International Game Theory Review*, 13(04), 443–459. 5

FISHER, J. C., AND I. E. HAFALIR (2016): "Matching with Aggregate Externalities," *Mathematical Social Sciences*, 81, 1–7. 1

FLEINER, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28(1), 103–126. 1, 15, 7, 7

GABAIX, X., AND D. LAIBSON (2006): "Shrouded Attributes, Consumer Myopia, and Information Suppression in Competitive Markets," *The Quarterly Journal of Economics*, 121(2), 505–540. 8

GALE, D., AND L. S. SHAPLEY (1962): "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 69(1), 9–15. 1, 3, 4, 15

GUDMUNDSSON, J., AND H. HABIS (2017): “Assignment games with externalities revisited,” *Economic Theory Bulletin*, 5(2), 247–257. 5

HAFALIR, I. E. (2008): “Stability of Marriage with Externalities,” *International Journal of Game Theory*, 37(3), 353–369. 5

HATFIELD, J., AND F. KOJIMA (2010): “Substitutes and Stability for Matching with Contracts,” *Journal of Economic Theory*, 145(5), 1704 – 1723. 1, 1, 15

HATFIELD, J., AND P. MILGROM (2005): “Matching with Contracts,” *American Economic Review*, 95(4), 913–935. 1, 1, 10, 4, 15, 7, 7, 7

HATFIELD, J. W., AND S. D. KOMINERS (2015): “Multilateral Matching,” *Journal of Economic Theory*, 156, 175 – 206. 6

——— (2016): “Hidden substitutes,” in *Proceedings of the 16th ACM Conference on Electronic Commerce*. 1

——— (2017): “Contract design and stability in many-to-many matching,” *Games and Economic Behavior*, 101, 78–97. 1, 1, 4, 7, 7

HATFIELD, J. W., S. D. KOMINERS, AND A. WESTKAMP (2017): “Stability, strategy-proofness, and cumulative offer mechanisms,” in *Proceedings of the 2017 ACM Conference on Economics and Computation*, pp. 673–674. ACM. 1

İNAL, H. (2015): “Core of Coalition Formation Games and Fixed-Point Methods,” *Social Choice and Welfare*, 45(4), 745–763. 1

KAMADA, Y., AND F. KOJIMA (2019): “Accommodating Various Policy Goals in Matching with Constraints,” Forthcoming at Japanese Economic Review. 4, 17

KELSO, A. S., AND V. P. CRAWFORD (1982): “Job Matching, Coalition Formation, and Gross Substitutes,” *Econometrica*, 50, 1483–1504. 1, 4, 3.1

KLAUS, B., AND F. KLIJN (2005): “Stable Matchings and Preferences of Couples,” *Journal of Economic Theory*, 121(1), 75–106. 1

KLAUS, B., AND M. WALZL (2009): “Stable many-to-many matchings with contracts,” *Journal of Mathematical Economics*, 45, 422 – 434. 1

KOJIMA, F., P. A. PATHAK, AND A. E. ROTH (2013): "Matching with Couples: Stability and Incentives in Large Markets," *The Quarterly Journal of Economics*, 128(4), 1585–1632. 1

KOTOWSKI, M. H. (2015): "A Note on Stability in One-to-One, Multi-Period Matching Markets," Faculty Research Working Paper RWP15-042, Harvard Kennedy School. 1

KURINO, M. (2014): "House Allocation with Overlapping Generations," *American Economic Journal: Microeconomics*, 6(1), 258–89. 1

LESHNO, J. (2019): "Existence of Stable Matching with Peer Effects in Large Markets," Work in Progress. 6

MCVITIE, D. G., AND L. B. WILSON (1970): "Stable marriage assignment for unequal sets," *BIT Numerical Mathematics*, 10(3), 295–309. 1

MUMCU, A., AND I. SAGLAM (2010): "Stable One-to-One Matchings with Externalities," *Mathematical Social Sciences*, 60(2), 154–159. 6

OSTROVSKY, M. (2008): "Stability in Supply Chain Networks," *American Economic Review*, 98(3), 897–923. 15, 7

PIGOU, A. (1932): *The Economics of Welfare*. American Mathematical Society. 1

PYCIA, M. (2008): "Non-Existence Result for Matching Games with Transfers," Working Paper. 6

——— (2012): "Stability and Preference Alignment in Matching and Coalition Formation," *Econometrica*, 80(1), 323–362. 1, 18

ROSTEK, M., AND N. YODER (2019a): "Matching with Complementary Contracts," Working paper. 7

——— (2019b): "Matching with Multilateral Contracts," Working paper. 7

ROTH, A., AND M. SOTOMAYOR (1990): *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*, vol. 18 of *Econometric Society Monographs*. Cambridge University Press, Cambridge England. 7

ROTH, A. E. (1984): "Stability and Polarization of Interests in Job Matching," *Econometrica*, 52(1), 47–57. 1, 10, 15, 7

ROTH, A. E., AND U. G. ROTHBLUM (1999): "Truncation strategies in matching markets—in search of advice for participants," *Econometrica*, 67(1), 21–43. 3.4

SALGADO-TORRES, A. (2011a): "A Solution Concept for Housing Market Problems with Externalities," *Economics Bulletin*, 31(1), 623–630. 5

——— (2011b): "Many-to-One Matching: Externalities and Stability," Working paper. 5

SASAKI, H., AND M. TODA (1996): "Two-Sided Matching Problems with Externalities," *Journal of Economic Theory*, 70(1), 93 – 108. 1

TEYTELBOYM, A. (2012): "Strong Stability in Networks and Matching Markets with Contracts," Working Paper. 1

UETAKE, K., AND Y. WATANABE (2012): "Entry by Merger: Estimates from a Two-Sided Matching Model with Externalities," Working Paper. 6

ÜNVER, M. U. (2010): "Dynamic Kidney Exchange," *Review of Economic Studies*, 77(1), 372–414. 1

WATSON, J. (2014): "Contractual Chains," Working Paper. 1

Appendix A: Law of Aggregate Demand and the Rural Hospitals Theorem

We provide a generalization of the law of aggregate demand (Hatfield and Milgrom, 2005) and size monotonicity (Alkan and Gale, 2003). In markets without externalities, this generalization is due to Fleiner (2003). For each contract $x \in \mathcal{X}$, there is a corresponding weight denoted by $w(x)$, which is strictly positive. The generalized law of aggregate demand requires that for agent $i \in \theta$ the total weight of contracts chosen from X conditional on μ is weakly smaller than the total weight of contracts chosen from X' conditional on μ' for any $X' \supseteq X$ and $\mu' \succeq^{\theta} \mu$. For a set of contracts $X \subseteq \mathcal{X}$, let $w(X) \equiv \sum_{x \in X} w(x)$. We provide a formal definition as follows.

Definition 7. Choice function c_i satisfies **the law of aggregate demand** if $i \in \theta$ and for any $X \subseteq X'$ and $\mu \preceq^{\theta} \mu'$ then $w(c_i(X|\mu)) \leq w(c_i(X'|\mu'))$.

Previous definitions in the matching literature are restricted to the settings without externalities, and assume that the weight on all contracts are exactly equal (with the only exception of Fleiner (2003)). Under this assumption, the generalized law of aggregate demand reduces to for any $X \subseteq X'$ and $\mu \subseteq \mathcal{X}$, $|c_i(X|\mu)| \leq |c_i(X'|\mu)|$. In terms of the demand metaphor of Hatfield and Milgrom (2005), all contracts are traded at price one. In contrast, we allow any prices.

We study how the weight of contracts changes for an agent in different stable matchings. We show that the weight remains the same regardless of the stable matching. This extends the rural hospitals theorem of Hatfield and Milgrom (2005) in two directions: We allow different contracts to have different weights and also preferences of an agent can depend on contracts signed by others.

Theorem 8. *(**Rural Hospital Theorem**) Suppose that choice functions satisfy substitutability, the law of aggregate demand for a weight function w, and that there exists a matching $\bar{\mu}^\theta$ such that for any $\mu \in \mathcal{M}^\theta$, $\bar{\mu}^\theta \geq^\theta \mu$ for side θ. Then, for any two stable matchings μ and μ', $w(\mu_i) = w(\mu'_i)$ for every agent i.*

Proof. First let us observe that since all weights are strictly positive, substitutability and the law of aggregate demand imply the irrelevance of rejected contracts. This is easy to see: Suppose that $X', X, \mu \subseteq \mathcal{X}$ are such that $c_i(X'_i|\mu) \subseteq X_i \subseteq X'_i$ for agent i. Then substitutability implies that $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$. Since weights are positive we get $w(c_i(X_i|\mu)) \geq w(c_i(X'_i|\mu))$. Now, since $X_i \subseteq X'_i$, the law of aggregate demand implies that $w(c_i(X_i|\mu)) \leq w(c_i(X'_i|\mu))$. Consequently, we need to have $w(c_i(X_i|\mu)) = w(c_i(X'_i|\mu))$. Since all weights are strictly positive and $c_i(X_i|\mu) \supseteq c_i(X'_i|\mu)$, we get $c_i(X_i|\mu) = c_i(X'_i|\mu)$, the desired conclusion.

Without loss of generality assume that $\theta = s$. Then, by Theorem 6, there exists a stable matching μ^*, which is seller-optimal and buyer-pessimal simultaneously. We show that for any stable matching μ, $w(\mu_i) = w(\mu^*_i)$. As it is shown in the proof of Theorem 6, f has two fixed points $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) such that $(A^{*s}, A^{*b}, \mu^*, \mu^*) \sqsupseteq (A^s, A^b, \mu, \mu)$. Therefore, $A^{*s} \supseteq A^s$, $A^{*b} \subseteq A^b$, $\mu^* \geq^{\mathcal{S}} \mu$ and $\mu^* \leq^{\mathcal{B}} \mu$. Now by the law of aggregate demand for any $i \in S$, $w(c_i(A^{*s}|\mu^*)) \geq w(c_i(A^s|\mu))$, which is equivalent to $w(\mu^*_i) \geq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When this is summed over all sellers, we get $w(\mu^*) \geq w(\mu)$. Similarly, for any $i \in B$, $w(c_i(A^{*b}|\mu^*)) \leq w(c_i(A^b|\mu))$, which is equivalent to $w(\mu^*_i) \leq w(\mu_i)$ since $(A^{*s}, A^{*b}, \mu^*, \mu^*)$ and (A^s, A^b, μ, μ) are fixed points of f. When summed over all buyers, this implies $w(\mu^*) \leq w(\mu)$. Therefore, $w(\mu^*) = w(\mu)$, moreover, all of the individual inequalities must hold as equalities implying that for any agent i, $w(\mu^*_i) = w(\mu_i)$. ■

Remark 2. The first part of the proof shows that the law of aggregate demand and the substitute condition imply the irrelevance of rejected contracts, thus extending an analogous result in Aygün and Sönmez (2013) to the setting with externalities. This part of the proof relies on the weights being strictly positive; the remainder of the proof does not. In particular, our proof thus establishes that the analogue of the rural hospitals theorem holds true for any profile of real weights, not necessarily positive, as long as we assume that that the choice functions satisfy the irrelevance of rejected contracts. In addition, under the assumptions of the theorem, an agent's choice from the same set conditional on two ranked matchings needs to be the same. Indeed, let $i \in \theta$ be an agent. Suppose that $X, \mu, \mu' \subseteq \mathcal{X}$ are such that $\mu \preceq^{\theta} \mu'$. Then, by substitutability, $c_i(X|\mu) \supseteq c_i(X|\mu')$. But the law of aggregate demand implies that $w(c_i(X|\mu)) \leq w(c_i(X|\mu'))$. Since all weights are positive, we get that $c_i(X|\mu) = c_i(X|\mu')$. This argument does not mean that we cannot have externalities because the choice conditional on two matchings that are not ranked with respect to $\succeq^{\theta}$ can still be different.

Appendix B: Group Stability

A set $X \subseteq \mathcal{X}$ **blocks** matching μ if $X \nsubseteq \mu$ and for all $i \in \mathcal{I}$ we have $X_i \subseteq c_i(\mu \cup X|\mu)$. Less formally, conditional on matching μ, every agent who is associated with a contract in X wants to sign all contracts in X associated with them. In this case, X is also called a **blocking set** for μ. A matching is **group stable** if it is individually rational matching and there is no blocking set of contracts. Without externalities, this stability concept has been used before (see, e.g., Roth, 1984 and Hatfield and Kominers (2017)).

Proposition 1. ***[Equivalence of Stability and Group Stability]*** *Suppose that choice functions satisfy substitutability. Then a matching is stable if, and only if, it is group stable.*

See Roth and Sotomayor (1990); Echenique and Oviedo (2006); Hatfield and Kominers (2017) for earlier developments of this equivalence when there are no externalities. In particular, Hatfield and Kominers (2017) prove the same result when there are no externalities. The same proof works in our setting as well. More precisely, the following lemma is enough to prove the proposition, which does not require the irrelevance of rejected contracts.

Lemma 4. *Suppose X blocks matching μ and choice functions satisfy substitutability. Then for every $x \in X \setminus \mu$, $\{x\}$ blocks μ.*

Proof. If X is a blocking set, then $X \subseteq C^{\mathcal{S}}(\mu \cup X|\mu) \cap C^{\mathcal{B}}(\mu \cup X|\mu)$. Take any $x \in X \setminus \mu$. Since choice function c_i satisfies substitutability, we have $r_i(\mu \cup \{x\}|\mu) \subseteq r_i(\mu \cup X|\mu)$ for every agent

i. This implies $x \in c_i(\mu \cup \{x\}|\mu)$ for every i, so $x \in C^{\mathcal{S}}(\mu \cup \{x\}|\mu) \cap C^{\mathcal{B}}(\mu \cup \{x\}|\mu)$. Therefore, $\{x\}$ is a blocking set for μ. ■

Appendix C: Proofs

After proving Theorem 1 and two auxiliary lemmas in Section 4.2, we then prove Theorem 4 using these two lemmas. The proof of this theorem precedes the proof of Theorem 2 because we use Theorem 4 in the proof of Theorem 2. We then come back to the usual order and prove Theorem 3 and Theorem 7.

Proof of Theorem 1

We first show the necessity that when C^θ satisfies substitutability, then, for each agent $i \in \theta$, there exists a list of preferences with the stated properties.

For any μ_{-i}, we can construct a list of preferences as follows. Let $x_1 \in c_i(\mathcal{X}|\mu_{-i})$, $x_2 \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$, $x_3 \in c_i(\mathcal{X} \setminus \{x_1, x_2\}|\mu_{-i})$, ... , $x_k \in c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{k-1}\}|\mu_{-i})$, and $c_i(\mathcal{X} \setminus \{x_1, \ldots, x_k\}|\mu_{-i}) = \emptyset$. This sequence creates an incomplete preference ranking over $\mathcal{X}_i \cup \{\emptyset\}$: $x_1 \succ^{\mu_{-i}} \ldots \succ^{\mu_{-i}} x_k \succ^{\mu_{-i}} \emptyset$. Consider all such preference rankings $(\succ_j^{\mu_{-i}})_{j \in \mathcal{J}}$. We need the following:

Claim: For any $X, \mu \subseteq \mathcal{X}$, $c_i(X|\mu_{-i}) = \bigcup_{j \in \mathcal{J}} \{x_j^{\mu_{-i}}\}$, where $x_j^{\mu_{-i}} = \max_{\succ_j^{\mu_{-i}}} (X \cup \{\emptyset\})$.[19]

Let $x \in c_i(X|\mu_{-i})$. We show that $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$. If $x \in c_i(\mathcal{X}|\mu_{-i})$, then $x = x_j^{\mu_{-i}}$ for some j. Suppose that $x \notin c_i(\mathcal{X}|\mu_{-i})$. If $c_i(X|\mu_{-i}) \supseteq c_i(\mathcal{X}|\mu_{-i})$, then the irrelevance of rejected contracts would imply $c_i(X|\mu_{-i}) = c_i(\mathcal{X}|\mu_{-i})$, which is a contradiction because $x \in c_i(X|\mu_{-i}) \setminus c_i(\mathcal{X}|\mu_{-i})$. Therefore, there exists $x_1 \in c_i(\mathcal{X}|\mu_{-i}) \setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_1 \notin X$. Consider preference rankings in $\mathcal{J}$ that have x_1 as their maximal contract. If $x \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$, then we are done since x_1 would be the maximal element of X with respect to a preference ranking since $x_1 \notin X$ and there would be a preference ranking in $\mathcal{J}$ such that $x_1 \succ x \succ \ldots$. Suppose that $x \notin c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$. By the irrelevance of rejected contracts, we cannot have $c_i(X|\mu_{-i}) \supseteq c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i})$. Therefore, there exists $x_2 \in c_i(\mathcal{X} \setminus \{x_1\}|\mu_{-i}) \setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_2 \notin X$. Repeat this argument. Suppose, for contradiction, that $x \notin c_i(\mathcal{X} \setminus \{x_1, \ldots, x_j\}|\mu_{-i})$ for all j. But there must exist some j^* for which $\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\} \subseteq X$. Then $x \in c_i(X|\mu_{-i})$ and standard substitutability imply that $x \in$

[19]For an analogue of this claim in the setting without externalities, see Chambers and Yenmez (2017).

$c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\}|\mu_{-i})$. This is a contradiction. Therefore, $x \in c_i(\mathcal{X} \setminus \{x_1, \ldots, x_{j^*}\}|\mu_{-i})$ for some j^*, which implies that $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$ because $\{x_1, \ldots, x_{j^*}\} \cap X = \varnothing$. Since $x \in c_i(X|\mu_{-i})$ implies $x = x_j^{\mu_{-i}}$ for some $j \in \mathcal{J}$, we get $c_i(X|\mu_{-i}) \subseteq \bigcup_{j\in\mathcal{J}} \{x_j^{\mu_{-i}}\}$.

Now let $x = x_j^{\mu_{-i}}$ for some j. This implies that for every $y \succ_j^{\mu_{-i}} x$, we have $y \notin X$. By construction, $x \in c_i(\mathcal{X} \setminus \bigcup_{y:y\succ_j^{\mu_{-i}} x} \{y\}|\mu_{-i})$. Standard substitutability and the fact that $\mathcal{X} \setminus \bigcup_{y:y\succ_j^{\mu_{-i}} x} \{y\} \supseteq X$ imply that $x \in c_i(X|\mu_{-i})$. This argument proves that $\bigcup_{j\in\mathcal{J}} \{x_j^{\mu_{-i}}\} \subseteq c_i(X|\mu_{-i})$. Therefore, $\bigcup_{j\in\mathcal{J}} \{x_j^{\mu_{-i}}\} = c_i(X|\mu_{-i})$, which concludes the proof of the claim. ■

Next we prove that, for any $\mu'_{-i} \succeq^\theta \mu_{-i} \succeq^\theta \varnothing$ and $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Take $\mu = \varnothing$ and construct the list of preferences $(\succ_j^{\varnothing})_{j\in\mathcal{J}}$ as above. For any $\mu_{-i} \succeq^\theta \varnothing$ and $X \subseteq \mathcal{X}$, $c_i(X|\mu_{-i}) \subseteq c_i(X|\varnothing)$ by monotone externalities. Thus, for each j, we can truncate the preference ranking $\succ_j^{\varnothing}$ to get a sequence as constructed above, call it $\succ_j^{\mu_{-i}}$.

For each $\mu \succeq^\theta \varnothing$, $c_i(X|\mu_{-i}) = \bigcup_{j\in\mathcal{J}} \{x_j^{\mu_{-i}}\}$ where $x_j^{\mu_{-i}} = \max_{\succ_j^{\mu_{-i}}} (X \cup \{\varnothing\})$ by construction. Furthermore, for any $\mu' \succeq^\theta \mu \succeq^\theta \varnothing$ and $X \subseteq \mathcal{X}$, $c_i(X|\mu'_{-i}) \subseteq c_i(X|\mu_{-i})$ by monotone externalities. Therefore, for any j, $\succ_j^{\mu'}$ and $\succ_j^{\mu_{-i}}$ are both truncations of $\succ_j^{\varnothing}$ such that $\succ_j^{\mu'_{-i}}$ is truncated at a weakly more-preferred contract than $\succ_j^{\mu_{-i}}$. Therefore, we get the conclusion that for any $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Finally, we show the sufficiency that when there exists a list of preferences with the desired properties, then C^θ satisfies substitutability. Standard substitutability follows from the decomposition result of Aizerman and Malishevski (1981). To show monotone externalities, suppose that $\mu' \succeq^\theta \mu \succeq^\theta \varnothing$, we need $R^\theta(X|\mu') \supseteq R^\theta(X|\mu)$ for every $X \subseteq \mathcal{X}$. Equivalently, we need that $r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i})$ for every $i \in \theta$ and $X \subseteq \mathcal{X}$. By the definition of $\succeq^\theta$, $\mu' \succeq^\theta \mu \succeq^\theta \varnothing$ implies $\mu'_{-i} \succeq^\theta \mu_{-i} \succeq^\theta \varnothing$ for every $i \in \theta$. By construction, there exists a list of preference rankings $(\succ_j^{\mu_{-i}})_{j\in\mathcal{J}}$ and $(\succ_j^{\mu'_{-i}})_{j\in\mathcal{J}}$ such that for every $j \in \mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Therefore, $r_i(X_i|\mu'_{-i}) \supseteq r_i(X_i|\mu_{-i})$ is satisfied. ■

Proof of Lemma 2

Function f is monotonic in $\sqsubseteq$ because for any $A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \preceq^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succeq^{\mathcal{B}} \tilde{\mu}^b$, substitutability implies that

$$\begin{aligned}
\mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^b) &\subseteq \mathcal{X}\backslash R^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b),\\
\mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^s) &\supseteq \mathcal{X}\backslash R^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s),
\end{aligned}$$

and consistency implies that

$$\begin{aligned} C^{\mathcal{S}}(A^s|\mu^s) &\leq^{\mathcal{S}} C^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s), \\ C^{\mathcal{B}}(A^b|\mu^b) &\geq^{\mathcal{B}} C^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b). \end{aligned}$$

Therefore, $(A^s, A^b, \mu^s, \mu^b) \sqsubseteq (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$ implies that $f(A^s, A^b, \mu^s, \mu^b) \sqsubseteq f(\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$. ■

Proof of Lemma 3

$A^s \cup A^b = A^s \cup \left[\mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s))\right] \supseteq A^s \cup [\mathcal{X} \setminus A^s] = \mathcal{X}$, so

$$A^s \cup A^b = \mathcal{X}.$$

Similarly, $A^s \cap A^b = A^s \cap \left[\mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s))\right] = A^s \setminus R^{\mathcal{S}}(A^s|\mu^s) = C^{\mathcal{S}}(A^s|\mu^s)$, which implies $C^{\mathcal{S}}(A^s|\mu^s) = A^s \cap A^b$. Analogously for buyers, $C^{\mathcal{B}}\left(A^b|\mu^b\right) = A^s \cap A^b$. Finally, $\mu^s = C^{\mathcal{S}}(A^s|\mu^s)$ and $\mu^b = C^{\mathcal{B}}\left(A^b|\mu^b\right)$ imply

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}(A^s|\mu^s).$$

■

Proof of Theorem 4

First, suppose that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. Claim 1 below shows that μ is a stable matching.

Claim 1. Suppose that the choice functions satisfy standard substitutability. Then matching μ is stable.

Proof. Suppose, for contradiction, that μ is not stable. Then there are three possibilities, all of which we proceed to rule out.

1. Matching μ is not individually rational for some seller j, that is $c_j(\mu|\mu) \subsetneq \mu_j$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, $C^{\mathcal{S}}(A^s|\mu) = \mu$ and $A^s \supseteq \mu$. But standard substitutability and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^s by agent j, that is $x \notin C^{\mathcal{S}}(A^s|\mu)$, a contradiction.

2. Matching μ is not individually rational for some buyer i, that is $c_i(\mu|\mu) \subsetneq \mu_i$. This is analogous to the previous case since f treats buyers and sellers symmetrically.

3. There exists a blocking pair $i \in \mathcal{B}$ and $j \in \mathcal{S}$ with contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\} |\mu) \cap c_j(\mu \cup \{x\} |\mu)$. Since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3, $A^s \cup A^b = \mathcal{X}$. Therefore, without loss of generality, assume that $x \in A^b$. Again, since $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f, by Lemma 3, $C^{\mathcal{B}}(A^b|\mu) = \mu$, which implies that $c_i(A^b|\mu) = \mu_i$. By the irrelevance of rejected contracts, for any set Y such that $A^b \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular, for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}|\mu) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \setminus \mu$.

To finish the proof of the theorem, we need to show that if matching μ is stable then there exist sets of contracts A^s, A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f. The following is useful in our construction of A^s and A^b.

Claim 2. Suppose that the choice functions satisfy standard substitutability. Then the function $M^\theta(\mu) \equiv \max\{X \subseteq \mathcal{X} | C^\theta(X|\mu) = \mu\}$, where the maximum is with respect to set inclusion, is well defined. Moreover, for any contract $x \notin M^\theta(\mu)$, $x \in C^\theta(M^\theta(\mu) \cup x|\mu)$.

Proof. If there are two sets M' and M'' such that $C^\theta(M'|\mu) = C^\theta(M''|\mu) = \mu$, then (by standard substitutability)

$$\begin{aligned} C^\theta(M' \cup M''|\mu) &= (M' \cup M'') \setminus R^\theta(M' \cup M''|\mu) = \left[M' \setminus R^\theta(M' \cup M''|\mu)\right] \cup \left[M'' \setminus R^\theta(M' \cup M''|\mu)\right] \\ &\subseteq \left[M' \setminus R^\theta(M'|\mu)\right] \cup \left[M'' \setminus R^\theta(M''|\mu)\right] = \mu. \end{aligned}$$

If $C^\theta(M' \cup M''|\mu)$ was a proper subset of μ, then the irrelevance of rejected contracts would imply that $C^\theta(M'|\mu) = C^\theta(M''|\mu) = C^\theta(M' \cup M''|\mu)$, which is a contradiction. Therefore, $M^\theta(\mu)$ is well defined. Let $x \notin M^\theta(\mu)$. If $x \notin C^\theta(M^\theta(\mu) \cup x|\mu)$, then $C^\theta(M^\theta(\mu) \cup x|\mu) = C^\theta(M^\theta(\mu)|\mu)$ by the irrelevance of rejected contracts. But this implies $C^\theta(M^\theta(\mu) \cup x|\mu) = \mu$, which contradicts maximality of $M^\theta(\mu)$. Hence, $x \in C^\theta(M^\theta(\mu) \cup x|\mu)$.

Claim 3. Suppose that matching μ is stable and the choice functions satisfy standard substitutability. Then there exist sets of contracts A^s and A^b such that $\left(A^s, A^b, \mu, \mu\right)$ is a fixed point of f.

Proof. By Claim 2, there exists the largest set $M^\theta(\mu) = \max\{X \subseteq \mathcal{X} | C^\theta(X|\mu) = \mu\}$. Let $A^s \equiv M^s(\mu)$ and $A^b \equiv \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)$. By construction of $M^b(\mu)$, $\mu = C^{\mathcal{S}}(A^s|\mu)$. Thus, we get $A^s \cap A^b = A^s \cap (\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)) = C^{\mathcal{S}}(A^s|\mu) = \mu$. To finish the proof, we need to show $\mu = C^{\mathcal{B}}(A^b|\mu)$ and $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu)$.

Note that $A^b = \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup C^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup \mu$. Therefore, $A^b \supseteq \mu$. If $Y \equiv C^{\mathcal{B}}(A^b|\mu) \neq \mu$, there are two cases, both of which contradict stability of μ. First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^{\mathcal{B}}(\mu|\mu) = Y$, implying that μ is not individually rational for some buyers, contradicting stability. Second, if $Y \not\subseteq \mu$, then there exists $y \in Y \setminus \mu$, and $y \in C^{\mathcal{B}}(\mu \cup \{y\}|\mu)$ by standard substitutability since $y \in C^{\mathcal{B}}(A^b|\mu)$ and $A^b \supseteq \mu \cup \{y\}$. But we also have that $y \in C^{\mathcal{S}}(A^s \cup \{y\}|\mu)$ by Claim 2. Then the agents associated with $\{y\}$ block μ, contradicting stability. Thus, the only case consistent with stability is $C^{\mathcal{B}}(A^b|\mu) = \mu$.

Finally, we show that $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus R^{\mathcal{B}}(\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)|\mu)$. Since $C^{\mathcal{B}}(A^b|\mu) = \mu$, then $\mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus (A^b \setminus \mu) = \mathcal{X} \setminus (((\mathcal{X} \setminus A^s) \cup \mu) \setminus \mu) = \mathcal{X} \setminus (\mathcal{X} \setminus A^s) = A^s$ and we have the result. ∎

Proof of Theorem 2

First, let us consider the first phase of the algorithm and check that $\mu^* \geq^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$. By the irrelevance of rejected contracts, we get $C^{\mathcal{S}}(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \geq^{\mathcal{S}} \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathcal{X} \supseteq \varnothing$ and consistency imply that

$$\mu_1 = C^{\mathcal{S}}(\mathcal{X}|\varnothing) \geq^{\mathcal{S}} C^{\mathcal{S}}(\varnothing|\varnothing) = \varnothing = \mu_0.$$

For the general case, $\mu_k \geq^{\mathcal{S}} \mu_{k-1}$ and $\mathcal{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^{\mathcal{S}}(\mathcal{X}|\mu_k) \geq^{\mathcal{S}} C^{\mathcal{S}}(\mu_k|\mu_{k-1}) = \mu_k.$$

Therefore, $\{\mu_k\}_{k\geq 1}$ is a monotone sequence with respect to the preorder $\geq^{\mathcal{S}}$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$; we take the minimum m satisfying this property and set $\mu^* = \mu_m$. Then,

$$C^{\mathcal{S}}(\mathcal{X}|\mu_m) = \mu_{m+1} = \mu_n \leq^{\mathcal{S}} \mu_m$$

where the monotonicity comparison follows because $\leq^{\mathcal{S}}$ is transitive.

It remains to show that the second phase converges and that the resulting matching is stable. It is easy to see that $f(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubseteq (\mathcal{X}, \varnothing, \mu^*, \varnothing)$ because $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \leq^{\mathcal{S}} \mu^*$ by construction and $C^{\mathcal{B}}(\varnothing|\varnothing) = \varnothing \geq^{\mathcal{B}} \varnothing$ by reflexivity of $\geq^{\mathcal{B}}$. By Lemma 2, f is monotone increasing, so we can repeatedly apply it to $f(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubseteq (\mathcal{X}, \varnothing, \mu^*, \varnothing)$ to get $f^k(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubseteq f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$

for every $k \geq 1$. We consider two separate possibilities. Suppose first that this sequence converges. Therefore, there exists k such that $f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing) = f^k(\mathcal{X}, \varnothing, \mu^*, \varnothing)$. As a result, $f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$ is a fixed point of f. Let $(\hat{A}^s, \hat{A}^b, \hat{\mu}^s, \hat{\mu}^b) \equiv f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$. By Lemma 3, $\hat{\mu}^s = \hat{\mu}^b = \hat{A}^s \cap \hat{A}^b$ and, by Theorem 4, $\hat{A}^s \cap \hat{A}^b$ is a stable matching.

Otherwise, if the sequence does not converge, there exists a subsequence $f^n(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsupseteq f^{n+1}(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsupseteq \ldots \sqsupseteq f^m(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsupseteq f^{m+1}(\mathcal{X}, \varnothing, \mu^*, \varnothing) = f^n(\mathcal{X}, \varnothing, \mu^*, \varnothing)$ because the number of contracts is finite. By transitivity of the preorder $\sqsupseteq$ and the previous inequality, we get $f^n(\mathcal{X}, \varnothing, \mu^*, \varnothing) = f^{m+1}(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsupseteq f^m(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsupseteq f^n(\mathcal{X}, \varnothing, \mu^*, \varnothing)$. Let $f^n(\mathcal{X}, \varnothing, \mu^*, \varnothing) = (A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ and $f^m(\mathcal{X}, \varnothing, \mu^*, \varnothing) = (A_2^s, A_2^b, \mu_2^s, \mu_2^b)$. By definition of $\sqsupseteq$, we get that $A_1^s = A_2^s$, $A_1^b = A_2^b$, $\mu_1^s \sim^s \mu_2^s$, and $\mu_1^b \sim^b \mu_2^b$. Now, by construction $C^{\mathcal{S}}(A_2^s|\mu_2^s) = \mu_1^s$ and by monotone externalities $C^{\mathcal{S}}(A_2^s|\mu_2^s) = C^{\mathcal{S}}(A_1^s|\mu_1^s)$, which imply that $C^{\mathcal{S}}(A_1^s|\mu_1^s) = \mu_1^s$. Similarly, we get that $C^{\mathcal{S}}(A_1^s|\mu_1^b) = \mu_1^b$. Furthermore, by monotone externalities, $\mathcal{X} \setminus R^{\mathcal{B}}(A_2^b|\mu_2^b) = \mathcal{X} \setminus R^{\mathcal{B}}(A_1^b|\mu_1^b)$ and, by construction, $\mathcal{X} \setminus R^{\mathcal{B}}(A_2^b|\mu_2^b) = A_1^b$, which imply $\mathcal{X} \setminus R^{\mathcal{B}}(A_1^b|\mu_1^b) = A_1^b$. Similarly, we get $\mathcal{X} \setminus R^{\mathcal{S}}(A_1^s|\mu_1^s) = A_1^s$. Therefore, $(A_1^s, A_1^b, \mu_1^s, \mu_1^b)$ is a fixed point of f. This shows that the sequence converges as in the previous paragraph, which is a contradiction. Therefore, there exists a stable matching. ∎

Proof of Theorem 3

Since choice function c_i has externalities, there exist $X, \mu, \mu' \subseteq \mathcal{X}$ such that $c_i(X|\mu') \neq c_i(X|\mu)$. This implies, without loss of generality, that there exists a contract $x \in X_i$ such that $x \in c_i(X_i|\mu'_{-i})$ and $x \notin c_i(X_i|\mu_{-i})$. We construct choice functions of agents other than i satisfying the stated properties such that no stable matching exists.

The choice functions of agents on side $-\theta$ exhibit no externalities. Furthermore, each agent chooses all the contracts in $\mu_{-i} \cup \mu'_{-i} \cup X_i$ that are associated with them whenever they are available. No other contracts are chosen. The choice functions of agents on side θ other than i depend on whether contract x is signed or not. When contract x is signed, each agent chooses contracts in μ_{-i} associated with them. When contract x is not signed, then each agent chooses contracts in μ'_{-i} associated with them. Otherwise, no contracts are chosen.

We first check that the properties in the statement of this result are satisfied. The agents on side $-\theta$ have substitutable choice functions that have no externalities by construction. Because they do not exhibit externalities, monotone externalities is trivially satisfied. Likewise, these choice functions also satisfy the irrelevance of rejected contracts. Now, consider the minimum consistent and separable preorder $\succeq^{\theta \setminus \{i\}}$ for the choice rules constructed for agents in $\theta \setminus \{i\}$.

The domain of this minimal preorder does not have a set that includes x (because x cannot be in any set constructed from any sequence of chosen contracts starting with the reference set $\varnothing$). For any reference set in this domain, the choice function remains the same, so monotone externalities is satisfied. Furthermore, by construction, substitutability and the irrelevance of rejected contracts are also satisfied.

Suppose, for contradiction, that there exists a stable matching Y. We consider two possibilities:

Case 1: Consider the case when $x \in Y$. If a contract in μ_{-i} is not signed, then the agents associated with the contract form a blocking pair. Thus, every contract in μ_{-i} must be signed, so $\mu_{-i} \subseteq Y_{-i}$. Furthermore, $Y_{-i} \setminus \mu_{-i}$ cannot have a contract as Y would not be individually rational for agents on side θ. Therefore, $\mu_{-i} = Y_{-i}$. Likewise, there cannot be any contract in $Y_i \setminus X_i$ because of individual rationality for agents on side $-\theta$. This implies that $Y_i \subseteq X_i$. If there exists a contract $x' \in c_i(X_i|\mu_{-i}) \setminus Y_i$, then agents associated with contract x' block Y because $x' \in c_i(Y_i \cup \{x'\}|\mu_{-i})$ by standard substitutability. Therefore, $Y_i \supseteq c_i(X_i|\mu_{-i})$. By the irrelevance of rejected contracts, $c_i(Y_i|\mu_{-i}) = c_i(X_i|\mu_{-i})$, which is a contradiction since $x \in Y_i = c_i(Y_i|\mu_{-i})$ by individual rationality of Y and $x \notin c_i(X_i|\mu_{-i})$ by construction.

Case 2: Consider the case when $x \notin Y$. As in the previous case, it is easy to see that $Y_{-i} = \mu'_{-i}$. Likewise, $Y_i \subseteq X_i$. Since $x \in c_i(X_i|\mu'_{-i})$ by construction, $x \in c_i(Y_i \cup \{x\}|\mu'_{-i})$ by standard substitutability. But this is a contradiction because $x \notin Y$ implies that the agents associated with contract x form a blocking pair.

Therefore, there exists no stable matching. ■

Proof of Theorem 7

By revealed preference, the Blair preorder for $C^{\mathcal{S}}$—which does not exhibit externalities—satisfies

$$C^{\mathcal{S}}(X|\mu) \geq^{\mathcal{S}} C^{*\mathcal{S}}(X|\mu)$$

for any $\mu, X \subseteq \mathcal{X}$.[20] Because μ^* is a $(C^{\mathcal{B}}, C^{*\mathcal{S}})$-stable matching, Theorem 4 gives us sets $A^s, A^b \subseteq \mathcal{X}$ such that (A^s, A^b, μ^*, μ^*) is a fixed point of the $\left(C^{*\mathcal{S}}, C^{\mathcal{B}}\right)$-analogue of function f

[20]The argument below applies also to $C^{\mathcal{S}}$ that have externalities as long as they admit a consistent preorder that satisfies the displayed property. As with the substitutes comparison, we can further weaken this property by imposing it only when $C^{\theta}(X|\mu) = \mu$; the weaker assumptions suffice as in the proof we apply this property to $C^{\mathcal{S}}$ and $\hat{C}^{\mathcal{S}}$ only when $C^{\mathcal{S}}(A^s|\mu) = \mu$.

from Lemma 2, defined as

$$\hat{f}\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^b), \mathcal{X}\backslash R^{*\mathcal{S}}(A^s|\mu^s), C^{*\mathcal{S}}\left(A^s|\mu^s\right), C^{\mathcal{B}}(A^b|\mu^b)\right).$$

The fixed point property, the contraction relation, and the above displayed property of $\succeq^{\mathcal{S}}$ imply that

$$(A^s, A^b, \mu^*, \mu^*) \sqsubseteq f(A^s, A^b, \mu^*, \mu^*),$$

where mapping f and preorder $\sqsubseteq$ are defined in Lemma 2. Indeed,

$$A^s = \mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^*)$$

by the fixed point property;

$$A^b = \mathcal{X}\backslash R^{*\mathcal{S}}(A^s|\mu^*) \supseteq \mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^*)$$

by the fixed point property and the contraction relation between $C^{\mathcal{S}}$ and $C^{*\mathcal{S}}$,

$$\mu^* = C^{*\mathcal{S}}\left(A^s|\mu^*\right) \preceq^{\mathcal{S}} C^{\mathcal{S}}\left(A^s|\mu^*\right)$$

by the fixed point property and the above displayed property of $\preceq^{\mathcal{S}}$;

$$\mu^* \succeq^{\mathcal{B}} \mu^* = C^{\mathcal{B}}(A^b|\mu^*)$$

by the fixed point property.

By Lemma 2, f is monotone increasing in preorder $\sqsubseteq$ and $f^{k-1}(A^s, A^b, \mu^*, \mu^*) \sqsubseteq f^k(A^s, A^b, \mu^*, \mu^*)$ for every $k \geq 1$. Since the number of contracts is finite, there exists k such that $f^{k-1}(A^s, A^b, \mu, \mu)$ is a fixed point of f as in the proof of Theorem 2. By Lemma 3, $f^{k-1}(A^s, A^b, \mu^*, \mu^*) = (\hat{A}^s, \hat{A}^b, \mu, \mu)$, and by Theorem 4, μ is a $(C^{\mathcal{B}}, C^{\mathcal{S}})$-stable matching. By construction, $\mu \succeq^{\mathcal{S}} \mu^*$ and $\mu^* \succeq^{\mathcal{B}} \mu$. ■

Appendix D: Additional Examples

In this section, we provide additional examples that satisfy substitutability.

Example 6. [Dynamic Matching][21] Firms and workers arrive to a two-sided matching market

[21] We would like to thank Maciej Kotowski for suggesting this example.

at times $t = 1, \ldots, T$. Workers who arrive at time t can wait and match at any time $t, t+1, \ldots, T$. At each time t a unique firm f_t arrives and either matches with one of the workers that is available at this time, or leaves unmatched. Firm f_t's ranking of workers is exogenously fixed but this firm's set of acceptable workers depends on the matches of firms $f_1, \ldots, f_{t-1}$: the higher firm f_1's worker in f_1's ranking, the more selective firm f_t becomes. If firm f_1 hires the same worker in two matchings, then the higher firm f_2's worker in f_2's ranking, the more selective firm f_t becomes, etc., lexicographically.

In this example, a consistent preorder for the firms is defined as follows: $\mu' \geq^{\theta} \mu$ if, and only if, for some firm f we have $\mu'(f) \succ_f \mu(f)$ and $\mu'(f') \succeq_{f'} \mu(f')$ for all firms f' matched before f. This preorder is separable, it is consistent with the choice functions, and the substitutability condition is satisfied as choosing out of larger (in inclusion sense) choice set conditional on a matching higher in this preorder, each firm continues to reject the worker it previously rejected. ■

Our theory applies to situations in which agents share profits, for instance because they work for the same firm, or have some insurance arrangements, or benefit from a public good financed by taxes on their private income. The following example illustrates a situation in which there is profit sharing.

Example 7. **[Profit Sharing]** Agents on one side of the market represent attorneys organized in law firms. Each attorney can work on up to $k \geq 0$ contracts with clients on the other side of the market; an attorney works on all contracts they sign and the attorney can also work on selected contracts signed by others in the same firm. Each contract allows an arbitrary number of attorneys to contribute; the profit an attorney makes from a contract does not depend on how many other attorneys contribute to it.[22] Each attorney prioritizes the contracts they work on, and the profit attorney i earns on a contract depends on whether it is the first, second, etc. contract in attorney i's priorities. We assume that each attorney must prioritize the contracts they sign over other contracts that they work on.

Attorneys choose what contracts to sign and what contracts to work on so as to maximize their profits: An attorney's profit is the sum of the profits from all the contracts they work on whether they signed it or not. We denote by $\lambda(x, i, \ell) \geq 0$ the profit that accrues to attorney i from working on contract x that they prioritize in position $\ell \in \{1, \ldots, k\}$. For simplicity, let us also assume that there are no indifferences.This example satisfies our assumptions provided

[22]This assumption and some of our other assumptions can be relaxed.

$\lambda(x,i,1) > \lambda(y,i,\ell)$ for all contracts x and y as long as attorney i is the signatory of contract x and $\ell > 1$.

Attorney choice functions satisfy substitutability if we define the preorder $\geq^\theta$ so that $\mu' \geq^\theta \mu$ if, and only if, $\max_{x \in \mu'(i)} \lambda(x,i,1) \geq \max_{x \in \mu(i)} \lambda(x,i,1)$ for all agents $i \in \theta$.[23] This preorder is separable and consistent with choice: When more contracts are available, the profitability of the best contract signed by each attorney goes up (irrespective of what contracts other attorneys sign). The substitutability condition holds for each attorney i: When more contracts are available and when the profitability of the best contract signed by other attorneys (and hence the outside option of attorney i) increases, the attorney continues to reject the contracts they previously rejected. ■

Our theory also applies to situations in which agents choose basic products with no regard to the choices of others but choose add-ons in a way that depends on others' choices of basic products. For instance, consider buyers who choose between Mac, PC, and Linux computers (and operating systems) in a way that does not depend on other buyers' choices and who take the hardware/operating system choices of others into account when buying productivity software.

Example 8. [Interoperability and Add-on Contracts] Suppose agents on one side (buyers) sign two types of contracts with sellers on the other side: for instance, agents might be signing primary contracts and add-on (or maintenance) contracts. These two classes of contracts are disjoint.[24] In line with the literature on add-on pricing, suppose that agents ignore the add-on contracts when deciding which primary contracts to sign (Gabaix and Laibson, 2006), and suppose that each agent signs at most one primary contract and that there are no externalities among primary contracts.[25]

We assume that no agent's choice of add-on contracts depends on the other agents' choices of add-on contracts, and we allow a buyer's choice among add-on contracts to depend on their and the other agents' choices of primary contracts in an arbitrary way as long as the buyer rejects weakly more (in the inclusion sense) add-on contracts out of X conditional on μ than

[23] We use the convention that the maximum over the empty set is $-\infty$.

[24] Similar examples can be written for hardware contracts and software contracts, or contracts on inputs and outputs.

[25] Formally, we assume that each buyer's choice among primary contracts does not depend on other agents' matches nor on the availability of add-on contracts. One reason that the agents ignore add-on contracts when signing primary contracts might be that the agents do not know which add-on contracts are available when signing the primary contracts as in Ellison (2005). We can relax the assumption that each agent signs at most one primary contract and assume instead that each agent's choice among primary contracts satisfies the standard substitutes assumption (see the next section).

they would reject out of X' conditional on μ' whenever $X \supseteq X'$ and the agent prefers their primary contracts in μ to those in μ'.

Buyer choice functions satisfy substitutability for the preorder $\geq^{\theta}$ such that $\mu' \geq^{\theta} \mu$ when each buyer prefers their primary contracts signed under μ' to those signed under μ. This preorder is separable and consistent: $\geq^{\theta}$ depends only on primary contracts, and each agent prefers to choose from larger choice sets over choosing from smaller choice sets. It is enough to check substitutability separately for the primary contracts and the add-on contracts: it holds for the primary contracts as the choice over them is not affected by externalities, and it holds for the add-on contracts as we explicitly assumed it. ■